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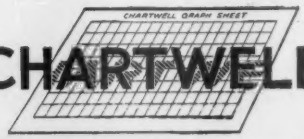

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ON COMPLETE RESIDUE SETS

By T. VIJAYARAGHAVAN (*Waltair*) and S. CHOWLA (*Ludhiana*)

[Received 8 February 1944; in revised form 12 December 1947]

1. THE following result is known:†

If q be an odd prime, $r_1, r_2,\ldots, r_q$ and $s_1, s_2,\ldots, s_q$ be two complete sets of residues (mod q), *then $r_1s_1, r_2s_2,\ldots, r_qs_q$ cannot be a complete set of residues* (mod q).

To prove the result we follow Pólya in supposing the contrary. We can take $r_q \equiv 0 \pmod q$ and then it is easy to deduce that $s_q \equiv 0 \pmod q$. We have then (to modulus q)

$$1.2.3...(q-1) \equiv r_1r_2...r_{q-1} \equiv s_1s_2...s_{q-1}$$
$$\equiv r_1s_1.r_2s_2...r_{q-1}s_{q-1} \equiv \{1.2.3...(q-1)\}^2$$

which is impossible since (by Wilson's theorem)

$$1.2.3...(q-1) \equiv -1 \pmod q.$$

We prove in this section that the above result is true not only for odd prime values of q but for all values of $q > 2$.

Suppose now that the result is not true for a composite value of q. It is shown below that there arises a contradiction. Let p be a prime divisor of q, and $q/p = N$. We see that r_ts_t is a multiple of p for precisely N values of t and that r_ts_t is prime to p for the remaining $q-N$ values of t. Since in each of the two sets $r_1, r_2,\ldots, r_q$ and $s_1, s_2,\ldots, s_q$ there are precisely $q-N$ numbers that are prime to p, we deduce at once that, whenever r_ts_t is a multiple of a prime number p that divides q, then r_t and s_t are both multiples of p. If we now make the further assumption that q is a multiple of p^2 as well, then we see that either r_ts_t is prime to p or is a multiple of p^2 and that therefore there is no value of t for which $r_ts_t \equiv p \pmod q$.

This contradiction proves the result when q is divisible by the square of a prime. It remains to prove the result when q is a product of two or more distinct primes. In this case we take an odd prime divisor p of q and consider the values of t for which r_ts_t is a multiple of N $(= p/q)$. There are precisely p such values of t; let these values

† A. Hurwitz, *Nouv. Ann.* Serie 3, 1 (1882), 389. See also G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. ii, chap. 8, problem 245, p. 158 and p. 379.

of t be $t_1, t_2, \ldots, t_p$. Now in each of the two sets $r_1, r_2, \ldots, r_q$ and $s_1, s_2, \ldots, s_q$ there are precisely p numbers that are multiples of N and precisely $q-p$ numbers that are not multiples of N. It follows that each of the numbers $r_{t_1}, r_{t_2}, \ldots, r_{t_p}$ is a multiple of N. Moreover, these p numbers in some order or other are congruent to $N, 2N, \ldots, pN \pmod q$ and are therefore incongruent $(\mathrm{mod}\, p)$. The same remarks apply to $s_{t_1}, s_{t_2}, \ldots, s_{t_p}$ and to $r_{t_1}s_{t_1}, r_{t_2}s_{t_2}, \ldots, r_{t_p}s_{t_p}$. But according to the result of A. Hurwitz this is not possible. This completes the proof when q is a product of two or more distinct primes. Hence we have the result:

If $r_1, r_2, \ldots, r_q$ and $s_1, s_2, \ldots, s_q$ are two complete residue sets $(\mathrm{mod}\, q)$, *where $q > 2$, then $r_1 s_1, r_2 s_2, \ldots, r_q s_q$ is not a complete residue set* $(\mathrm{mod}\, q)$.

2. The main result of this note is given in this section.

We consider the following problem. Suppose that n is a positive integer, $\phi(n) = h$, and $r_1, r_2, \ldots, r_h$ are all prime to n and incongruent $(\mathrm{mod}\, n)$. Such a set may be called a complete *primitive* residue set $(\mathrm{mod}\, n)$. Suppose now that $r_1, r_2, \ldots, r_h$ and $s_1, s_2, \ldots, s_h$ are two such sets. Can it happen that the product set $r_1 s_1, r_2 s_2, \ldots, r_h s_h$ is also a complete primitive residue set? It is easy to see from the proof of the result of A. Hurwitz that the product set cannot be a complete primitive residue set if n is a prime number > 2; it is easy to verify that the same is the case if $n = 4, 6, 9$, etc. But we see from the following table that for some other values of n the product set can be a complete primitive residue set provided that the first two sets are suitably ordered.

	$n = 2$	$n = 8$	$n = 12$	$n = 15$
r_t	1	1, 3, 5, 7	1, 5, 7, 11	1, 2, 4, 7, 8, 11, 13, 14
s_t	1	1, 5, 7, 3	1, 7, 11, 5	1, 4, 14, 2, 11, 7, 13, 8
$r_t s_t$ reduced $(\mathrm{mod}\, n)$	1	1, 7, 3, 5	1, 11, 5, 7	1, 8, 11, 14, 13, 2, 4, 7

It turns out that there is a neat answer to the query: 'Which numbers have the property considered above?' The answer is given by the following

THEOREM. *If $n = 2$ or has no primitive root, then there exist suitable complete primitive residue sets $r_1, r_2, \ldots, r_h$ and $s_1, s_2, \ldots, s_h$ such that $r_1 s_1, r_2 s_2, \ldots, r_h s_h$ too is a complete primitive residue set.*

Remark. If $n > 2$ and has a primitive root g, then it is easy to show that n has not the property under consideration. For otherwise we should have to modulus n

$$g.g^2...g^h \equiv r_1r_2...r_h \equiv s_1s_2...s_h \equiv r_1s_1r_2s_2...r_hs_h \equiv (g.g^2...g^h)^2,$$

which is a contradiction since $n > 2$ and

$$g.g^2...g^h \equiv g^{wh+\frac{1}{2}h} \equiv g^{\frac{1}{2}h} \equiv -1 \pmod{n},$$

where $w = \frac{1}{2}h$ is an integer.

LEMMA. *If m and n are prime to each other and the conclusion of the theorem is true for m and n, then it is true for mn.*

Let $\phi(m) = h$, $\phi(n) = k$, and $r_1, r_2,..., r_h, s_1, s_2,..., s_h$ and $r_1s_1, r_2s_2,..., r_hs_h$ be three complete primitive residue sets (mod m), and let $\rho_1, \rho_2,..., \rho_k, \sigma_1, \sigma_2,..., \sigma_k$, and $\rho_1\sigma_1, \rho_2\sigma_2,..., \rho_k\sigma_k$ be three such sets (mod n). Let $\{\alpha, \beta\}$ denote the residue class x (mod mn), where x is such that $x \equiv \alpha \pmod{m}$, $x \equiv \beta \pmod{n}$, and let $R_1, R_2,..., R_{hk}$ be a complete primitive residue set (mod mn). If $R_a = \{r_b, \rho_c\}$, then we take $S_a = \{s_b, \sigma_c\}$ ($a = 1, 2, 3,..., hk$). It is easy to verify that $S_1, S_2,..., S_{hk}$ and $R_1S_1, R_2S_2,..., R_{hk}S_{hk}$ are two complete primitive residue sets (mod mn), and this proves the lemma.

The theorem is first proved for values of n that belong to a set S, where S consists precisely of the five following forms:

(1) $n = 2^\lambda$, where $\lambda \neq 2$;
(2) $n = 2^\lambda m$, where $\lambda \geqslant 2$ and m is a power of any odd prime;
(3) $n = p^\lambda q^\mu$, where p and q are any pair of distinct odd primes;
(4) $n = 4M$, where M is any member of the form (3) mentioned just above;
(5) $n = p^\lambda q^\mu r^\nu$, where p, q, r are any three distinct odd primes.

It may be remarked here that, if a number n has no primitive root, then either it is a member of S or can be represented as a product of two or more mutually prime members of S. In view of the lemma already proved it follows immediately that the theorem of this note is completely proved when it has been proved for all values of n that belong to S.

(I) $n = p^\lambda q^\mu$. Let g be a primitive root of p^λ, $\phi(p^\lambda) = 2M$, g' a primitive root of q^μ and $\phi(q^\mu) = 2N$. We denote by $\{\alpha, \beta\}$ the residue class x which is such that

$$x \equiv g^\alpha \pmod{p^\lambda}, \qquad x \equiv g'^\beta \pmod{q^\mu}.$$

It should be noticed that by giving to α the values $0, 1, 2, \ldots, 2M-1$ and to β the values $0, 1, 2, \ldots, 2N-1$ we get all the $4MN$ primitive residue classes $(\mathrm{mod}\, p^\lambda q^\mu)$. Also

$$\{\alpha, \beta\} = \{\alpha+2M, \beta\} = \{\alpha, \beta+2N\}$$

for every pair of values α, β; the converse is also true, i.e. if

$$\{\alpha, \beta\} = \{\alpha', \beta'\},$$

then $\alpha \equiv \alpha' \pmod{2M}$ and $\beta \equiv \beta' \pmod{2N}$. Finally, if $x = \{\alpha, \beta\}$ and $y = \{\alpha', \beta'\}$, then $xy = \{\alpha+\alpha', \beta+\beta'\}$ for all $\alpha, \beta, \alpha', \beta'$. These properties enable us to solve the problem under consideration. Let $r_1, r_2, \ldots, r_h$ (where $h = 4MN$) be a complete set of primitive residues $(\mathrm{mod}\, n)$. We show below how a complete set of primitive residue classes $s_1, s_2, \ldots, s_h$ can be chosen in such a way that $r_1 s_1, r_2 s_2, \ldots, r_h s_h$ is also a complete primitive residue set.

If $r_t = \{\alpha, \beta\}$ $(1 \leqslant \alpha \leqslant M; 1 \leqslant \beta \leqslant N)$,
then s_t is to be taken equal to $\{\alpha, \beta\}$;

if $r_t = \{\alpha, \beta\}$ $(M < \alpha \leqslant 2M;\ 1 \leqslant \beta \leqslant N)$,
then s_t is to be taken equal to $\{\alpha, \beta-1\}$;

if $r_t = \{\alpha, \beta\}$ $(M \leqslant \alpha < 2M;\ N < \beta \leqslant 2N)$,
then s_t is to be taken equal to $\{\alpha+1, \beta-1\}$;

if $r_t = \{\alpha, \beta\}$ $(0 \leqslant \alpha < M;\ N < \beta \leqslant 2N)$,
then s_t is to be taken equal to $\{\alpha+1, \beta\}$.

It is easy to verify that, if $r_1, r_2, \ldots, r_h$ be a complete primitive residue set, the same is true of $s_1, s_2, \ldots, s_h$ and also of $r_1 s_1, r_2 s_2, \ldots, r_h s_h$.

The proofs are as follows:

(i) *for the numbers* r_t. From the first two lines of the above scheme we see that α takes $2M$ incongruent values $(\mathrm{mod}\, 2M)$ when

$$1 \leqslant \beta \leqslant N;$$

from the third and fourth lines we see that α takes $2M$ incongruent values $(\mathrm{mod}\, 2M)$ when $N < \beta \leqslant 2N$;

(ii) *for the numbers* s_t. From the first and fourth lines of the scheme we see that $s_t = \{\alpha, \beta\}$, where $1 \leqslant \alpha \leqslant M$ and β takes $2N$ incongruent values $(\mathrm{mod}\, 2N)$; from the second and third lines of the scheme we see that $s_t = \{\alpha, \beta\}$, where $M < \alpha \leqslant 2M$ and β takes $2N$ incongruent values $(\mathrm{mod}\, 2N)$;

(iii) *for the numbers* $r_t s_t$. Here we have $r_t s_t = \{\alpha, \beta\}$, where

in the first line, α takes all even values (mod $2M$),
β takes all even values (mod $2N$);

in the second line, α takes all even values (mod $2M$),
β takes all odd values (mod $2N$);

in the third line, α takes all odd values (mod $2M$),
β takes all odd values (mod $2N$);

in the fourth line, α takes all odd values (mod $2M$),
β takes all even values (mod $2N$).

In all the cases (i), (ii), (iii) we get $4MN$ numbers $\{\alpha, \beta\}$, where α runs through $2M$ incongruent values (mod $2M$) and β runs through $2N$ incongruent values (mod $2N$). Thus we have proved that the three sets r_t, s_t, and $r_t s_t$ $(1 \leqslant t \leqslant h)$ are complete primitive residue sets.

We can present the choices in the above scheme more briefly in a tabular form. [In the table given below the 'type' to which $r_t s_t$ belongs is indicated; if α is even and β is odd we shall say that $r_t s$ belongs to the type $+-$. The three other types $++$, $-+$, $--$ are similarly defined.]

$$n = p^\lambda q^\mu; \qquad r_i = \{\alpha, \beta\}$$

α	β	s_t	$r_t s_t$
$1 \leqslant \alpha \leqslant M$	$1 \leqslant \beta \leqslant N$	$\{\alpha, \beta\}$	$++$
$M < \alpha \leqslant 2M$	$1 \leqslant \beta \leqslant N$	$\{\alpha, \beta-1\}$	$+-$
$M \leqslant \alpha < 2M$	$N+1 \leqslant \beta \leqslant 2N$	$\{\alpha+1, \beta-1\}$	$--$
$0 \leqslant \alpha < M$	$N < \beta \leqslant 2N$	$\{\alpha+1, \beta\}$	$-+$

An even more brief representation of the table would be

r_t	11	21	2′2	1′2
s_t	11	21′	22′	12
$r_t s_t$	$++$	$+-$	$--$	$-+$

(II) $n = 4q^\mu$. This case is disposed of in exactly the same way as $n = p^\lambda q^\mu$ since the number 4 has the primitive root 3. The case $2^\lambda q^\mu$, where $\lambda > 2$, is discussed a little farther down.

(III) $n = 2^\lambda$ $(\lambda > 2)$. This case is disposed of in exactly the same way as $p^\lambda q^\mu$ for the following reason. Any primitive residue class (mod 2^λ) can be represented as $\{\alpha, \beta\}$, where $\{\alpha, \beta\}$ represents the residue class x, if and only if $x \equiv 5^\alpha (-1)^\beta \pmod{n}$.

We get all the residue classes by giving to α the values 0, 1, 2,...,

$2^{\lambda-2}-1$, and to β the values 0 and 1. This representation has all the properties mentioned earlier in connexion with the case $n = p^\lambda q^\mu$. We give below the details of the choice of $s_1, s_2, \ldots, s_h$, where

$$h = 2^{\lambda-1} = 4M.$$

r_t	$-5, -5^2, \ldots, -5^M$	$-5^{M+1}, \ldots, -5^{2M}$	$5^M, 5^{M+1}, \ldots, 5^{2M-1}$	$1, 5, 5^2, \ldots, 5^{M-1}$
s_t	$-5, -5^2, \ldots, -5^M$	$5^{M+1}, \ldots, 5^{2M}$	$-5^{M+1}, -5^{M+2}, \ldots, -5^{2M}$	$5, 5^2, 5^3, \ldots, 5^M$
$r_t s_t$	$5^{2\alpha}$	$-5^{2\alpha}$	$-5^{2\alpha+1}$	$5^{2\alpha+1}$
	$(1 \leqslant \alpha \leqslant M)$	$(M < \alpha \leqslant 2M)$	$(M \leqslant \alpha < 2M)$	$(0 \leqslant \alpha < M)$

(IV) $n = p^\lambda q^\mu r^\nu$. Let g, g', g'' be respectively primitive roots of p^λ, q^μ, r^ν. We denote by $\{\alpha, \beta, \gamma\}$ the residue class $x \pmod n$, where

$$x \equiv g^\alpha \pmod{p^\lambda}, \quad x \equiv g'^\beta \pmod{q^\mu}, \quad x \equiv g''^\gamma \pmod{r^\nu}.$$

The choice of s_t is made according to the following table:

r_t	111	211	222	122	2′21	1′21	2′12	1′12
s_t	111	21′1	22′2′	122′	22′1′	121′	21′2	112
$r_t s_t$	$+++$	$+-+$	$+--$	$++-$	$---$	$-+-$	$--+$	$-++$

A more explicit version of this table would be

$$n = p^\lambda q^\mu r^\nu, \qquad \phi(p^\lambda) = 2M, \qquad \phi(q^\mu) = 2N,$$
$$\phi(r^\nu) = 2L, \qquad r_t = \{\alpha, \beta, \gamma\}$$

α	β	γ	s_t	$r_t s_t$
$1 \leqslant \alpha \leqslant M$	$1 \leqslant \beta \leqslant N$	$1 \leqslant \gamma \leqslant L$	$\{\alpha, \beta, \gamma\}$	$+++$
$M < \alpha \leqslant 2M$	$1 \leqslant \beta \leqslant N$	$1 \leqslant \gamma \leqslant L$	$\{\alpha, \beta-1, \gamma\}$	$+-+$
$M < \alpha \leqslant 2M$	$N < \beta \leqslant 2N$	$L < \gamma \leqslant 2L$	$\{\alpha, \beta-1, \gamma-1\}$	$+--$
$1 \leqslant \alpha \leqslant M$	$N < \beta \leqslant 2N$	$L < \gamma \leqslant 2L$	$\{\alpha, \beta, \gamma-1\}$	$++-$
$M \leqslant \alpha < 2M$	$N < \beta \leqslant 2N$	$1 \leqslant \gamma \leqslant L$	$\{\alpha+1, \beta-1, \gamma-1\}$	$---$
$0 \leqslant \alpha < M$	$N < \beta \leqslant 2N$	$1 \leqslant \gamma \leqslant L$	$\{\alpha+1, \beta, \gamma-1\}$	$-+-$
$M \leqslant \alpha < 2M$	$1 \leqslant \beta \leqslant N$	$L < \gamma \leqslant 2L$	$\{\alpha+1, \beta-1, \gamma\}$	$--+$
$0 \leqslant \alpha < M$	$1 \leqslant \beta \leqslant N$	$L < \gamma \leqslant 2L$	$\{\alpha+1, \beta, \gamma\}$	$-++$

(V) $n = 4q^\mu r^\nu$. This case is disposed of like the previous case since the number 4 has the primitive root 3.

(VI) $n = 2^\lambda r^\nu$ $(\lambda > 2)$. This case also is covered by the discussion in the case $n = p^\lambda q^\mu r^\nu$, for the residue class $x \pmod{2^\lambda r^\nu}$ can be represented by $\{\alpha, \beta, \gamma\}$, where α, β, γ are such that

$$x \equiv 5^\alpha (-1)^\beta \pmod{2^\lambda}, \qquad x \equiv g^\gamma \pmod{r^\nu},$$

g being a primitive root of r^ν. This completes all the cases included in the set S, and, as pointed out already, the proof of the theorem is now plain.

SUMMARY

It is known that, if $q > 2$ and q is prime, then there do not exist two complete residue sets $r_1, r_2, \ldots, r_q$ and $s_1, s_2, \ldots, s_q$ such that $r_1 s_1, r_2 s_2, \ldots, r_q s_q$ also is a complete residue set (mod q). It is pointed out in this note that the same conclusion holds not only for prime values of q but also for all numbers $q > 2$. The main result of the note is the theorem

THEOREM. *If $n > 2$ and $\phi(n) = h$, then there exist complete primitive residue sets $r_1, r_2, \ldots, r_h$ and $s_1, s_2, \ldots, s_h$ such that $r_1 s_1, r_2 s_2, \ldots, r_h s_h$ too is a complete primitive residue set if and only if n has no primitive root.*

A THEOREM ON INFINITE PRODUCTS OF TRANSFINITE CARDINAL NUMBERS

By F. BAGEMIHL (*Rochester, N.Y.*)

[Received 25 June 1947]

THE following is a well-known theorem concerning sums of alephs:*

(I) *If α is a transfinite limit-number, and $\{\sigma_\xi\}_{\xi<\alpha}$ is an increasing sequence of ordinal numbers such that $\lim_{\xi<\alpha}\sigma_\xi = \lambda$, then*

$$\sum_{\xi<\alpha}\aleph_{\sigma_\xi} = \sum_{\xi<\alpha}\aleph_\lambda = \aleph_\lambda.$$

Tarski† posed the problem of deciding the truth or falsity of the analogous proposition for products of alephs:

(II) *If α is a transfinite limit-number, and $\{\sigma_\xi\}_{\xi<\alpha}$ is an increasing sequence of ordinal numbers such that $\lim_{\xi<\alpha}\sigma_\xi = \lambda$, then*

$$\prod_{\xi<\alpha}\aleph_{\sigma_\xi} = \prod_{\xi<\alpha}\aleph_\lambda = \aleph_\lambda^{\bar{\alpha}}.$$

If certain restrictions are placed on the numbers σ_ξ of the sequence $\{\sigma_\xi\}_{\xi<\alpha}$ or on α, then (II) is known to be true. Thus, Tarski‡ proved (III) and (IV):

(III) *If α is a transfinite limit-number, then*

$$\prod_{\xi<\alpha}\aleph_\xi = \aleph_\alpha^{\bar{\alpha}}.$$

Here $\{\sigma_\xi\}_{\xi<\alpha}$ is the special sequence $\{\xi\}_{\xi<\alpha}$.

(IV) *Proposition* (II) *holds if we further assume that*

$$\alpha = \omega^\delta \quad (\delta > 0).$$

Hausdorff obtained the next result:§

(V) *Proposition* (II) *is true if we make the additional hypothesis on α that $\bar{\alpha} = \bar{\rho}$ whenever $\alpha = \beta+\rho$ and $\rho \neq 0$.*

* Small Greek letters, with or without subscripts, will be used throughout to denote ordinal numbers. For an explanation of the notation and terminology, and the theorems which I presuppose, consult (**1**), (**2**), (**3**).

† (**3**), 13.

‡ (**3**), 11–12, 6–7.

§ Communicated to Tarski; stated in (**3**), 14.

Finally, the following result, which places no such limitations on α or $\{\sigma_\xi\}_{\xi<\alpha}$, but which is weaker than (II), has been stated (without proof) by Tarski (4):

(VI) *If α is a transfinite limit-number, and $\{\sigma_\xi\}_{\xi<\alpha}$ is an increasing sequence of ordinal numbers such that $\lim_{\xi<\alpha}\sigma_\xi = \lambda$, then there exists a remainder ρ of α ($\rho \neq 0$) such that*

$$\prod_{\xi<\alpha} \aleph_{\sigma_\xi} = \aleph_\lambda^\rho.$$

I shall prove that (II) is valid if, roughly speaking, the numbers coming at the end of the sequence $\{\sigma_\xi\}$ in question are not too far apart. More precisely:

THEOREM. *Let α be a transfinite limit-number, and let its normal form be*

$$\alpha = \omega^{\delta_1}+\omega^{\delta_2}+\ldots+\omega^{\delta_n} = \sum_{1\leqslant k\leqslant n} \omega^{\delta_k}, \tag{1}$$

where n is a natural number and

$$\delta_1 \geqslant \delta_2 \geqslant \ldots \geqslant \delta_n > 0. \tag{2}$$

Let $\{\sigma_\xi\}_{\xi<\alpha}$ be an increasing sequence of ordinals such that $\lim_{\xi<\alpha}\sigma_\xi = \lambda$, and subject, if $n > 1$, to the following condition:

if $\beta = \sum_{1\leqslant k\leqslant n-1} \omega^{\delta_k}$, *and* $\lim_{\xi<\beta}\sigma_\xi = \eta$, *then* $\overline{-\eta+\lambda} \leqslant \overline{\omega^{\delta_n}}$.* (3)

Then

$$\prod_{\xi<\alpha} \aleph_{\sigma_\xi} = \aleph_\lambda^{\overline{\alpha}}.$$

Proof.† Assume that the theorem is false.

Let α be the smallest transfinite limit-number which satisfies the hypotheses of the theorem and is such that

$$\prod_{\xi<\alpha} \aleph_{\sigma_\xi} \neq \aleph_\lambda^{\overline{\alpha}}. \tag{4}$$

Then

$$n > 1; \tag{5}$$

otherwise (1) and (4) would entail a contradiction of (IV). From (5), (2), and the definition of β, we infer that

$$\beta \text{ is a transfinite limit-number;} \tag{6}$$

and (2) and

$$\alpha = \beta+\omega^{\delta_n} \tag{7}$$

imply that

$$\omega^{\delta_n} \leqslant \beta < \alpha. \tag{8}$$

* This last inequality actually implies equality.

† The method is essentially that used by Tarski to establish (III).

The relation $$\bar{\beta} = \bar{\alpha} \tag{9}$$

then results from (7), (8), and the fact that, if $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are alephs such that $\mathfrak{a} < \mathfrak{c}$ and $\mathfrak{b} < \mathfrak{c}$, then $\mathfrak{a}+\mathfrak{b} < \mathfrak{c}$.

Because of (6), (8), (4), and (9),

$$\prod_{\xi<\beta} \aleph_{\sigma_\xi} = \aleph_\eta^{\bar{\beta}} = \aleph_\eta^{\bar{\alpha}}, \tag{10}$$

and, by (IV), we have

$$\prod_{\xi<\omega^{\delta_\pi}} \aleph_{\sigma_{\beta+\xi}} = \aleph_\lambda^{\overline{\omega^{\delta_\pi}}}. \tag{11}$$

It follows from (10) and (11) that

$$\prod_{\xi<\alpha} \aleph_{\sigma_\xi} = \Big(\prod_{\xi<\beta} \aleph_{\sigma_\xi}\Big)\Big(\prod_{\xi<\omega^{\delta_\pi}} \aleph_{\sigma_{\beta+\xi}}\Big) = \aleph_\eta^{\bar{\alpha}} \cdot \aleph_\lambda^{\overline{\omega^{\delta_\pi}}}. \tag{12}$$

By (3) and (8), $$\overline{-\eta+\lambda} \leqslant \overline{\omega^{\delta_\pi}} \leqslant \bar{\alpha}. \tag{13}$$

Now, there is the following theorem due to Tarski:*

If $\bar{\rho} \leqslant \mathfrak{a}$, where $\mathfrak{a}$ is a transfinite cardinal number, then

$$\aleph_{\theta+\rho}^{\mathfrak{a}} = \aleph_\theta^{\mathfrak{a}} \cdot \aleph_{\theta+\rho}^{\bar{\rho}}.$$

If we take $\theta = \eta$, $\rho = -\eta+\lambda$, and $\mathfrak{a} = \bar{\alpha}$, and observe (13), this theorem yields

$$\aleph_\lambda^{\bar{\alpha}} = \aleph_{\eta+(-\eta+\lambda)}^{\bar{\alpha}} = \aleph_\eta^{\bar{\alpha}} \cdot \aleph_{\eta+(-\eta+\lambda)}^{\overline{-\eta+\lambda}} \leqslant \aleph_\eta^{\bar{\alpha}} \cdot \aleph_\lambda^{\overline{\omega^{\delta_\pi}}}; \tag{14}$$

and comparing (12) and (14) we see that

$$\prod_{\xi<\alpha} \aleph_{\sigma_\xi} \geqslant \aleph_\lambda^{\bar{\alpha}}.$$

But certainly $$\prod_{\xi<\alpha} \aleph_{\sigma_\xi} \leqslant \prod_{\xi<\alpha} \aleph_\lambda = \aleph_\lambda^{\alpha},$$

so that we must have $$\prod_{\xi<\alpha} \aleph_{\sigma_\xi} = \aleph_\lambda^{\bar{\alpha}}.$$

This contradicts (4), however. Hence our theorem is true.

COROLLARY. *If α is a transfinite limit-number, and μ is an arbitrary ordinal number, then*

$$\prod_{\xi<\alpha} \aleph_{\mu+\xi} = \aleph_{\mu+\alpha}^{\bar{\alpha}}. \tag{15}$$

For, if we take in our theorem $\sigma_\xi = \mu+\xi$ for $\xi < \alpha$, the additional condition on $\{\sigma_\xi\}_{\xi<\alpha}$ is automatically satisfied.

If, in particular, we set $\mu = 0$ in (15), the latter becomes Tarski's theorem (III).

* (3), 2.

The theorems stated in this paper can easily be re-worded so as to be valid when the α in them is taken to denote a suitable isolated ordinal instead of a transfinite limit-ordinal.

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SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

By D. D. KOSAMBI (*Bombay*)

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1. We consider here the system of equations

$$\frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} + H^i_{\alpha\beta}(t^\mu, x^k, p^r_\nu) = 0 \tag{1.1}$$

$$(i,j,\ldots = 1,2,\ldots,n;\ \alpha,\beta,\ldots = 1,2,\ldots,m;\ p^i_\alpha = \partial x^i/\partial t^\alpha).$$

For $m < n$ we can say that we are exploring the x-space by means of m-dimensional sections, or hypersurfaces. For $m \geqslant n$ no such interpretation will do; in either case we explore not only the x-space but the t-space as well, and the interrelation of the two is not always clear. This again limits the formal treatment that has been followed. Specially interesting cases are $m = 1$, usually treated without parameter transformations; $m = n-1$, $m = n$, and $n = 1$, for any order of partial differential equations.

The transformation group under which (1.1) are assumed to be invariant is

$$\bar{x} \longleftrightarrow x, \quad \left|\frac{\partial \bar{x}^i}{\partial x^j}\right| \neq 0; \qquad \bar{t} \longleftrightarrow t, \quad \left|\frac{\partial \bar{t}^\alpha}{\partial t^\beta}\right| \neq 0; \tag{1.2}$$

where, naturally, either set of variables can be considered by themselves. That is, we might have x-transformations alone with tensors having only Latin indices, or t-transformations alone with tensors having only Greek indices. These cases, however, are immediately derivable from the formulae we give in the general case, by suppression of one or the other type of index. Both sorts of indices are summed when repeated in subscript and superscript.

The tensor p^i_α is of mixed type, with $\bar{p}^i_\alpha = p^r_\nu(\partial \bar{x}^i/\partial x^r)(\partial t^\nu/\partial \bar{t}^\alpha)$. The condition that (1.1) be tensor-invariant under both sorts of transformations is

$$-\bar{H}^i_{\alpha\beta} = -H^r_{\nu\sigma}\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial t^\nu}{\partial \bar{t}^\alpha}\frac{\partial t^\sigma}{\partial \bar{t}^\beta} + \frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^s}p^r_\sigma p^s_\nu \frac{\partial t^\sigma}{\partial \bar{t}^\alpha}\frac{\partial t^\nu}{\partial \bar{t}^\beta} + \frac{\partial^2 t^\nu}{\partial \bar{t}^\alpha \partial \bar{t}^\beta}\frac{\partial \bar{x}^i}{\partial x^r}p^r_\nu. \tag{1.3}$$

To save space I adopt the notation:

$$\left.\begin{gathered}\frac{\partial F^{\cdot\cdot}_{\cdot\cdot}}{\partial t^\alpha} = F^{\cdot\cdot}_{\cdot\cdot,\alpha}, \qquad \frac{\partial F^{\cdot\cdot}_{\cdot\cdot}}{\partial x^i} = F^{\cdot\cdot}_{\cdot\cdot,i}, \qquad \frac{\partial F^{\cdot\cdot}_{\cdot\cdot}}{\partial p^i_\alpha} = F^{\cdot\cdot}_{\cdot\cdot}|^\alpha_i \\ \partial_\alpha F^{\cdot\cdot}_{\cdot\cdot} = F^{\cdot\cdot}_{\cdot\cdot,\alpha} + p^r_\alpha F^{\cdot\cdot}_{\cdot\cdot,r} - H^r_{\nu\alpha}F^{\cdot\cdot}_{\cdot\cdot}|^\nu_r\end{gathered}\right\}. \tag{1.4}$$

The operator ∂_α amounts therefore to total differentiation with respect to t along a 'path', but is clearly not tensorial. The operator $\partial/\partial p^i_\alpha$ is tensorial but does not correspond to any sort of proper covariant differentiation. However, arguments parallel in all details to those used for the ordinary path-spaces (**1**) show that the differential operator applied to an x-vector must be of type

$$D_\alpha u^i = \partial_\alpha u^i + \gamma^i_{\alpha r} u^r,$$

and such an operation can be vectorial in the Greek index as well, but will not by itself apply to any vector with Greek indices. The deficiency must then be made up by adding another summation term with an affine connexion (assumed symmetric) in the Greek index. That is, the general operator must be of form

$$D_\alpha T^{i\mu\cdot}_{j\nu\cdot} = \partial_\alpha T^{i\mu\cdot}_{j\nu\cdot} + \gamma^i_{\alpha r} T^{r\mu\cdot}_{j\nu\cdot} - \gamma^r_{\alpha j} T^{i\mu\cdot}_{r\nu\cdot} + \Gamma^\mu_{\alpha\rho} T^{i\rho\cdot}_{j\nu\cdot} - \Gamma^\rho_{\alpha\nu} T^{i\mu\cdot}_{j\rho\cdot}. \quad (1.5)$$

This, however, is for a tensor of *weight* zero, i.e. one which is not multiplied by any power of the Jacobian of the transformation. For tensors of weight q under x-transformations and p under t-transformations it is necessary (**2**) to add the terms

$$-(q\gamma^r_{\alpha r} + p\Gamma^\rho_{\alpha\rho}) T^{i\mu\cdot}_{j\nu\cdot}$$

to the right-hand side of (1.5). The corresponding changes have to be made in the formulae that follow.

The laws of transformation for the two sets of connexion coefficients are then determined completely in the form

$$\gamma^s_{\sigma r} \frac{\partial \bar{x}^i}{\partial x^s} \frac{\partial t^\sigma}{\partial \bar{t}^\alpha} = \bar{\gamma}^i_{\alpha s} \frac{\partial \bar{x}^s}{\partial x^r} + p^s_\sigma \frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^s} \frac{\partial t^\sigma}{\partial \bar{t}^\alpha}, \quad (1.6)$$

and

$$\bar{\Gamma}^\sigma_{\alpha\beta} \frac{\partial t^\nu}{\partial \bar{t}^\sigma} = \Gamma^\nu_{\sigma r} \frac{\partial t^\sigma}{\partial \bar{t}^\alpha} \frac{\partial t^r}{\partial \bar{t}^\beta} + \frac{\partial^2 t^\nu}{\partial \bar{t}^\alpha \partial \bar{t}^\beta}. \quad (1.7)$$

For the path-space to exist, i.e. for exploration by means of the paths to be actually possible, the integrability conditions of (1.1) must be satisfied. These are

$$H^i_{\alpha\beta} - H^i_{\beta\alpha} = 0,$$

$$\partial_\nu H^i_{\alpha\beta} - \partial_\beta H^i_{\alpha\nu} \equiv H^i_{\alpha\beta,\nu} + p^r_\nu H^i_{\alpha\beta,r} - H^r_{\sigma\nu} H^i_{\alpha\beta}|^\sigma_r - H^i_{\alpha\nu,\beta} - p^r_\beta H^i_{\alpha\nu,r} + H^r_{\sigma\beta} H^i_{\alpha\nu}|^\sigma_r = 0, \quad (1.8)$$

and we shall assume that both of them are fulfilled.

2. It is very convenient to have an intrinsic determination for the two sets of connexion coefficients $\gamma^i_{\alpha j}$ and $\Gamma^\alpha_{\beta\nu}$, which reduces the calculation of the differential invariants and of the relationship

between them; all invariants of the space can be expressed in terms of any intrinsic set, plus two more 'invariants of the connexion', for any two sets of connexion coefficients of the space can only differ by tensor terms, according to (1.6) and (1.7). Further, it will be seen that the very convenient method of determining the connexion from the equations of variation is not applicable without modification; as a matter of fact, it is not applicable at all to higher orders than the second. So we resort to a direct determination, from the transformation laws of the $H^i_{\alpha\beta}$.

Differentiating both sides of (1.3) with respect to $\bar{p}^\mu_j$ we get

$$-\bar{H}^i_{\alpha\beta}|^\mu_j = -H^r_{\nu\sigma}|^\lambda_k \frac{\partial \bar{t}^\mu}{\partial t^\lambda}\frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial t^\nu}{\partial \bar{t}^\alpha}\frac{\partial t^\sigma}{\partial \bar{t}^\beta}+$$

$$+\frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^s}\left\{\delta^\mu_\alpha \frac{\partial x^r}{\partial \bar{x}^j}\frac{\partial t^\nu}{\partial \bar{t}^\beta}p^s_\nu + \delta^\mu_\beta \frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial t^\sigma}{\partial \bar{t}^\alpha}p^r_\sigma\right\}+\frac{\partial^2 t^\nu}{\partial \bar{t}^\alpha \partial \bar{t}^\beta}\frac{\partial \bar{t}^\mu}{\partial t^\nu}\delta^i_j;$$

and

$$-\bar{H}^i_{\alpha\beta}|^\mu_j|^\epsilon_k = -H^r_{\nu\sigma}|^\lambda_s|^\eta_p \frac{\partial \bar{t}^\mu}{\partial t^\lambda}\frac{\partial \bar{t}^\epsilon}{\partial t^\eta}\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^\nu}{\partial \bar{x}^k}\frac{\partial t^\nu}{\partial \bar{t}^\alpha}\frac{\partial t^\sigma}{\partial \bar{t}^\beta}+$$

$$+\frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^s}\{\delta^\mu_\alpha \delta^\epsilon_\beta + \delta^\mu_\beta \delta^\epsilon_\alpha\}\frac{\partial x^r}{\partial \bar{x}^j}\frac{\partial x^s}{\partial \bar{x}^k}. \qquad (2.1)$$

These enable us to solve the main difficulty of getting just one additive term from x-transformations or one from t-transformations into the connexion-coefficient laws (1.6) and (1.7). We see at once that admissible choices of the intrinsic connexion are

$$\left.\begin{aligned}\gamma^i_{\alpha j} &= \frac{1}{m(m+1)}H^i_{\rho\sigma}|^\rho_j|^\sigma_r p^r_\alpha \\ \Gamma^\nu_{\alpha\beta} &= \frac{1}{n}\{H^r_{\alpha\beta}|^\nu_r|^\sigma_j p^j_\sigma - H^r_{\alpha\beta}|^\nu_r\}\end{aligned}\right\}. \qquad (2.2)$$

The transformation laws (1.6) show that $H^i_{\alpha\beta}|^\nu_j|^\mu_k|^\sigma_l$ and $\Gamma^\nu_{\alpha\beta}|^\rho_i$ are tensors, as also $\gamma^i_{\alpha j}|^\rho_k|^\nu_l$. But our intrinsic determination shows immediately with (2.1) that the last two are a consequence here of the former being tensorial. The tensor character of the original equations is shown by putting them in the form

$$D_\beta p^i_\alpha + A^i_{\alpha\beta} = 0, \qquad A^i_{\alpha\beta} = H^i_{\alpha\beta} - p^r_\alpha \gamma^i_{\beta r} + p^i_\rho \Gamma^\rho_{\alpha\beta}. \qquad (2.3)$$

The second integrability condition may be put into the form

$$(D_\beta D_\alpha - D_\alpha D_\beta)p^i_\nu = p^r_\nu\{\partial_\alpha \gamma^i_{\beta r} - \partial_\beta \gamma^i_{\alpha r} + \gamma^k_{\beta r}\gamma^i_{\alpha k} - \gamma^k_{\alpha r}\gamma^i_{\beta k}\} -$$

$$-p^i_\sigma\{\partial_\beta \Gamma^\sigma_{\alpha\nu} - \partial_\alpha \Gamma^\sigma_{\beta\nu} + \Gamma^\sigma_{\beta\mu}\Gamma^\mu_{\alpha\nu} - \Gamma^\sigma_{\alpha\mu}\Gamma^\mu_{\beta\nu}\}. \qquad (2.4)$$

For the connexion chosen, the first integrability condition reduces to

$$A^i_{\alpha\beta}-A^i_{\beta\alpha}=0.$$

The equations of variation are of two sorts, for one may give a vector variation λ^i to x, and another μ^α to t. These are, for x,

$$\partial_\alpha\partial_\beta\lambda^i+\partial_\nu\lambda^r H^i_{\alpha\beta}|^\nu_r+\lambda^r H^i_{\alpha\beta,r}=0, \tag{2.5}$$

which can be put in the invariantive form

$$D_\alpha D_\beta\lambda^i+D_\nu\lambda^r A^{i\nu}_{r\alpha\beta}+\lambda^r P^i_{r\alpha\beta}=0; \tag{2.6}$$

and the t-variation equation (obtained by putting $\bar t^\alpha=t^\alpha+\epsilon\mu^\alpha$, and neglecting ϵ^2 and higher powers in the formal expansion) which has the tensor form

$$p^i_\nu D_\beta D_\alpha\mu^\nu+D_\nu\mu^\sigma Q^{i\nu}_{\alpha\beta\sigma}+\mu^\sigma R^i_{\alpha\beta\sigma}=0. \tag{2.7}$$

The tensors that appear in these equations of variation are

$$\left.\begin{aligned}
A^{i\nu}_{j\alpha\beta}&=\delta^i_j\Gamma^\nu_{\alpha\beta}-\delta^\nu_\beta\gamma^i_{\alpha j}-\delta^\nu_\alpha\gamma^i_{\beta j}+H^i_{\alpha\beta}|^\nu_j\\
P^i_{j\alpha\beta}&=H^i_{\alpha\beta,j}-H^i_{\alpha\beta}|^\sigma_r\gamma^r_{\sigma j}+\gamma^r_{\beta j}\gamma^i_{\alpha r}-\partial_\beta\gamma^i_{\alpha j}\\
Q^{i\nu}_{\alpha\beta\sigma}&=H^i_{\alpha\beta}|^\nu_j p^j_\sigma-\delta^\nu_\beta H^i_{\alpha\sigma}-\delta^\nu_\alpha H^i_{\beta\sigma}+(\Gamma^\nu_{\alpha\beta}\delta^\rho_\sigma-\Gamma^\rho_{\alpha\sigma}\delta^\nu_\beta-\Gamma^\rho_{\beta\sigma}\delta^\nu_\alpha)p^i_\rho\\
R^i_{\alpha\beta\sigma}&=p^i_\nu(\Gamma^\nu_{\rho\sigma}\Gamma^\rho_{\alpha\beta}+\Gamma^\nu_{\alpha\rho}\Gamma^\rho_{\beta\sigma}-\partial_\beta\Gamma^\nu_{\alpha\sigma})-H^i_{\alpha\beta,\sigma}-\Gamma^\rho_{\nu\sigma}Q^{i\nu}_{\alpha\beta\rho}
\end{aligned}\right\}. \tag{2.8}$$

The two sets of equations of variation (2.6), (2.7) are not independent. Putting $\lambda^i=p^i_\rho\mu^\rho$ immediately links the vector x-variation to the vector t-variation, thus changing (2.6) into (2.7). This leads to two identities independent of the connexion coefficients:

$$\left.\begin{aligned}
&\text{(a)}\quad Q^{i\nu}_{\alpha\beta\sigma}=\delta^\nu_\alpha D_\beta p^i_\sigma+\delta^\nu_\beta D_\alpha p^i_\sigma+A^{i\nu}_{r\alpha\beta}p^r_\sigma\\
&\qquad\qquad=A^{i\nu}_{r\alpha\beta}p^r_\sigma-\delta^\nu_\alpha A^i_{\sigma\beta}-\delta^\nu_\beta A^i_{\sigma\alpha}\\
&\text{(b)}\quad R^i_{\alpha\beta\sigma}=p^r_\sigma P^i_{r\alpha\beta}-A^{i\rho}_{r\alpha\beta}A^r_{\sigma\rho}-D_\beta A^i_{\sigma\alpha}
\end{aligned}\right\}. \tag{2.9}$$

3. To obtain the complete set of differential invariants, it remains to alternate the operations. The first gives the covariant x-derivative proper

$$(D_\alpha T^{i\sigma})|^\beta_j-D_\alpha T^{i\sigma}|^\beta_j=\delta^\beta_\alpha T^{i\sigma}_{|j}-A^{r\beta}_{j\rho\alpha}T^{i\sigma}|^\rho_r+T^{i\rho}\Gamma^\sigma_{\rho\alpha}|^\beta_j+T^{r\sigma}B^{i\beta}_{\alpha rj}, \tag{3.1}$$

where

$$T^{i\sigma}_{|j}=T^{i\sigma}_{,j}-\gamma^r_{\rho j}T^{i\sigma}|^\rho_r+\frac{1}{m}T^{r\sigma}\gamma^i_{\nu r}|^\nu_j, \tag{3.2}$$

and

$$B^{i\beta}_{\alpha jk}=\gamma^i_{\alpha j}|^\beta_k-\frac{1}{m}\delta^\beta_\alpha\gamma^i_{\rho j}|^\rho_k. \tag{3.3}$$

For the special connexion adopted (3.3) leads to

$$B^{i\beta}_{\alpha jk} = \frac{1}{m(m+1)}\left\{H^i_{\rho\sigma}|^\rho_j|^\sigma_r|^\beta_k p^r_\alpha - \frac{1}{m}\delta^\beta_\alpha H^i_{\sigma\rho}|^\rho_j|^\sigma_r|^\mu_k p^r_\mu\right\}, \tag{3.4}$$

so that we get no new intrinsic differential invariant. The operator in (3.2), however, is the one remaining operator needed to complete the set of tensor-operations, while a term $-p\gamma^r_{\rho r}|^\rho_j T^{i\sigma}$ must be added on the right-hand side if $T^{i\sigma}$ is a relative invariant of x-weight p.

Further

$$T^{i\sigma}|^\nu_{j\,k} - T^{i\sigma}|^\nu_{k|j} = \frac{1}{m}T^{r\sigma}\gamma^i_{\rho r}|^\rho_j|^\nu_k - T^{i\sigma}|^\rho_r\{B^{r\nu}_{\rho jk} + \delta^\nu_\rho T^r_{jk}\}, \tag{3.5}$$

where $B^{r\nu}_{\rho jk}$ is as in the previous alternation, and

$$T^i_{jk} = \frac{1}{m}\{\gamma^i_{\rho j}|^\rho_k - \gamma^i_{\rho k}|^\rho_j\} = \frac{1}{m^2(m+1)}\{H^i_{\rho\sigma}|^\rho_j|^\sigma_r|^\alpha_k p^r_\alpha - H^i_{\rho\sigma}|^\rho_k|^\sigma_r|^\alpha_j p^r_\alpha\}. \tag{3.6}$$

The covariant derivative alternates with D_α as

$$(D_\alpha T^{i\sigma})_{|j} - D_\alpha T^{i\sigma}_{|j} = T^{i\rho}\Gamma^\sigma_{\alpha\rho|j} + T^{r\sigma}C^i_{\alpha rj} - T^{i\sigma}|^\rho_r P^r_{j\rho\alpha}, \tag{3.7}$$

where $P^i_{j\alpha\beta}$ occurs in the equations of variation, $\Gamma^\sigma_{\alpha\rho|j}$ is a new differential invariant (as seen directly from the transformation laws), and the remaining coefficient is given by

$$C^i_{\alpha jk} = \gamma^i_{\alpha j,k} - \gamma^r_{\beta k}\gamma^i_{\alpha j}|^\beta_r - \frac{1}{m}\{\partial_\alpha \gamma^i_{\beta j}|^\beta_k + \gamma^i_{\alpha r}\gamma^r_{\beta j}|^\beta_k - \gamma^r_{\alpha j}\gamma^i_{\beta r}|^\beta_k - \gamma^r_{\alpha k}\gamma^i_{\beta j}|^\beta_r\}. \tag{3.8}$$

The D operators alternate to

$$(D_\beta D_\alpha - D_\alpha D_\beta)T^{i\sigma} = T^{r\sigma}\{P^i_{r\alpha\beta} - P^i_{r\beta\alpha}\} + T^{i\rho}\{\partial_\beta \Gamma^\sigma_{\alpha\rho} - \partial_\alpha \Gamma^\sigma_{\beta\rho} + \\ + \Gamma^\sigma_{\beta\mu}\Gamma^\mu_{\alpha\rho} - \Gamma^\sigma_{\alpha\mu}\Gamma^\mu_{\beta\rho}\}. \tag{3.9}$$

The last coefficient on the right has the form of the Riemann–Christoffel curvature tensor, and the relationship

$$R^i_{\alpha\beta\sigma} - R^i_{\beta\alpha\sigma} = p^i_\nu\{\partial_\alpha \Gamma^\nu_{\beta\sigma} - \partial_\beta \Gamma^\nu_{\alpha\sigma} + \Gamma^\nu_{\alpha\rho}\Gamma^\rho_{\beta\sigma} - \Gamma^\nu_{\beta\rho}\Gamma^\rho_{\alpha\sigma}\} = p^i_\nu R^\nu_{\sigma\alpha\beta}. \tag{3.10}$$

This does not, of course, suffice to determine the new tensor. In any of these alternations, a lower index occasions only a minus sign in the corresponding summation term, and there is one summation term for each such index, so that there would be no difficulty in calculating the alternation for any number of indices. For (3.9) applied to p^i_α we have

$$(D_\beta D_\alpha - D_\alpha D_\beta)p^i_\nu = p^r_\nu\{P^i_{r\alpha\beta} - P^i_{r\beta\alpha}\} + R^i_{\alpha\beta\nu} - R^i_{\beta\alpha\nu}. \tag{3.11}$$

This, however, is identically the form of the condition of integrability (2.4).

The final alternation is of the pure covariant derivative

$$T^{i\sigma}_{|k|j}-T^{i\sigma}_{|j|k}=T^{r}_{kj}T^{i\sigma}_{|r}+(C^{r}_{\rho kj}-C^{r}_{\rho jk}+D_{\rho}T^{r}_{kj})T^{i\sigma}|^{\rho}_{r}+$$
$$+T^{r\sigma}\{T^{i}_{rj|k}-T^{i}_{rk|j}+T^{i}_{lr}T^{l}_{kj}+T^{i}_{lj}T^{l}_{rk}+T^{i}_{lk}T^{l}_{jr}\}+$$
$$+(C^{r}_{\rho jk}-C^{r}_{\rho kj}+D_{\rho}T^{r}_{kj})|^{\rho}_{r}+B^{i\mu}_{\rho jl}B^{l\rho}_{\mu kr}-B^{i\mu}_{\rho kl}B^{ls}_{\mu jr}. \quad (3.12)$$

4. *The basis for the field of differential invariants is seen from the derivation of the previous section to consist of the set*

$$\left.\begin{matrix} p^{i}_{\alpha}, & A^{i\nu}_{j\alpha\beta}, & P^{i}_{j\alpha\beta}, & H^{i}_{\alpha\beta}|^{\nu}_{j}|^{\mu}_{k}|^{\rho}_{l} \\ & \Gamma^{\nu}_{\alpha\beta|j}, & C^{i}_{\alpha jk}, & R^{\nu}_{\sigma\alpha\beta} \end{matrix}\right\}. \quad (4.1)$$

To these, of course, must be added two 'invariants of the connexion' as we have used a specialized connexion to simplify the derivation.

It is instructive to consider the case of a single parameter t, i.e. $m=1$. Here, $H^{i}_{\alpha\beta}=\alpha^{i}$, $p^{i}_{\alpha}=\dot{x}^{i}$, and the paths become

$$\ddot{x}^{i}+\alpha^{i}(t,x,\dot{x})=0. \quad (4.2)$$

This case has been treated in detail elsewhere (**1**) for x-transformations only, taking t as an absolute parameter. The specialized connexion coefficients become

$$\gamma^{i}_{\alpha j}=\gamma^{i}_{j}=\tfrac{1}{2}\alpha^{i}_{;j;r}\dot{x}^{r}, \qquad \Gamma^{\nu}_{\mu\sigma}=\Gamma=\frac{1}{n}\{\alpha^{r}_{;r;j}\dot{x}^{j}-\alpha^{r}_{;r}\}. \quad (4.3)$$

The semicolon in subscript denotes $\partial/\partial\dot{x}$. With t an absolute parameter, I have used the (also permissible) values $2\gamma^{i}_{j}=\alpha^{i}_{;j}$, and the terms in Γ can, naturally, be suppressed. For α^{i} homogeneous of degree two in $\dot{x}$ which includes the Riemannian, Finsler, and affine connexion cases, the two values of γ^{i}_{j} coincide, and $\Gamma\equiv 0$. It may be noted that $\dot{x}^{i}$ is a covariant vector in t, or alternatively of t-weight unity, so that

$$D\dot{x}^{i}=-\alpha^{i}+\gamma^{i}_{r}\dot{x}^{r}-\Gamma\dot{x}^{i},$$

and the investigation corresponds in essence to that of *projective* connexions. The invariants of (4.1) become

$$A^{i\nu}_{j\mu\sigma}=A^{i}_{j}=\delta^{i}_{j}\Gamma-2\gamma^{i}_{j}+\alpha^{i}_{;j}$$
$$=\alpha^{i}_{;j}-\alpha^{i}_{;j;k}\dot{x}^{k}+\frac{1}{n}\delta^{i}_{j}\{\alpha^{r}_{;r}-\alpha^{r}_{;r;k}\dot{x}^{k}\}, \quad (4.4)$$

$$P^{i}_{j\lambda\mu}=P^{i}_{j}=\alpha^{i}_{;j}-\alpha^{i}_{;r}\gamma^{r}_{j}+\gamma^{i}_{r}\gamma^{r}_{j}-\frac{d}{dt}\gamma^{i}_{j}. \quad (4.5)$$

This (with a change of sign) is the curvature tensor obtained for systems of ordinary differential equations.

$$\left.\begin{aligned}\Gamma^{\nu}_{\lambda\mu|j} &= \Gamma_{\nu|j} = \Gamma_{\nu;j} - \gamma^{r}_{j}\Gamma_{\nu r} \\ H^{i}_{\mu\nu|k}{}^{\rho}{}_{|\ell}{}^{\sigma}{}^{\lambda} &= \alpha^{i}_{\nu;k;\ell}\end{aligned}\right\}. \tag{4.6}$$

$$C^{i}_{\alpha jk} = C^{i}_{jk} = \gamma^{i}_{j,k} - \gamma^{i}_{k}\gamma^{k}_{r} - \frac{d}{dt}\gamma^{i}_{j;k} - \gamma^{i}_{r}\gamma^{r}_{j;k} + \gamma^{r}_{j}\gamma^{i}_{r;k} + \gamma^{r}_{k}\gamma^{i}_{j;r}. \tag{4.7}$$

This last is equal, for $\gamma^{i}_{j} = \frac{1}{2}\alpha^{i}_{;j}$, to

$$P^{i}_{j;k} - \tfrac{2}{3}(P^{i}_{j;k} - P^{i}_{k;j}),$$

and the term actually does occur in the alternation of corresponding differential operators. Now it must be kept in mind that in applying tensor differentiation in the present case, even for $m = 1$, the silent Greek indices must be taken into account, their net effect being an additive term $k\Gamma T^{\cdot\cdot}_{\cdot\cdot}$ with k equal to the total number of contravariant indices minus the total number of covariant indices. The operations themselves are

$$\left.\begin{aligned} D_{\alpha}T^{\cdot\cdot}_{\cdot\cdot} = DT^{\cdot\cdot}_{\cdot\cdot} &= \frac{d}{dt}T^{\cdot\cdot}_{\cdot\cdot} + \gamma^{i}_{r}T^{r\,\cdot\cdot}_{\cdot\cdot\,\ldots} - \gamma^{r}_{\nu}T^{\cdot\cdot}_{\nu\cdot\,\ldots} \\ T^{\cdot\cdot}_{\cdot\cdot|j} = T^{\cdot\cdot}_{\cdot\cdot;j} &- \gamma^{r}_{j}T^{\cdot\cdot}_{\cdot\cdot;r} + \gamma^{i}_{r;j}T^{r\,\cdot\cdot}_{\cdot\cdot\,\ldots} - \gamma^{r}_{k;j}T^{\cdot\cdot}_{r\cdot}\end{aligned}\right\}. \tag{4.8}$$

Here no additional terms for relative invariance have been shown. There remains the operation $\partial/\partial\dot{x}^{i}$, and one may anticipate the next section to introduce another operator (linearly dependent on the three above):

$$T^{\cdot\cdot}_{\cdot\cdot|0} = \frac{\partial T^{\cdot\cdot}_{\cdot\cdot}}{\partial t} + \dot{x}^{r}\Gamma T^{\cdot\cdot}_{\cdot\cdot;r} + k\Gamma T^{\cdot\cdot}_{\cdot\cdot}\,. \tag{4.9}$$

In the case where t-transformations are not allowed, i.e. $\bar{t} = t$, the simplest determination of the connexion is $\gamma^{i}_{j} = \frac{1}{2}\alpha^{i}_{;j}$, for this annuls $A^{i\nu}_{j\alpha\beta}$ and simplifies all succeeding formulae.

5. However, for the general case $m > 1$, there are reasons for seeking other intrinsic connexions than those given. It will be recalled that the possibility of using connexion coefficients that vanished along a path was a very useful method in ordinary path-space. In the general case we have the result:

It is possible to choose a set of coordinates that makes $\gamma^{i}_{\alpha j} = 0$ along a sufficiently small portion of any given arbitrary path if and only if

$$P^{i}_{j\alpha\beta} - P^{i}_{j\beta\alpha} \equiv \partial_{\alpha}\gamma^{i}_{\beta j} - \partial_{\beta}\gamma^{i}_{\alpha j} + \gamma^{i}_{\alpha r}\gamma^{r}_{\beta j} - \gamma^{i}_{\beta r}\gamma^{r}_{\alpha j} = 0 \tag{5.1}$$

over the entire space.

In fact the transformation law of the $\gamma^i_{\alpha j}$ is the same as for γ^i_j in ordinary path-space except for multiplication by a coefficient $\partial t^\mu/\partial \bar{t}^\nu$ for the Greek index, as for a covariant vector. If, now, we have to make the connexion coefficients vanish along a path-base, we have to find a family of n independent vectors $V^i_{(j)}$ which are parallelly displaced along the base, and then set them equal to the proper transformation coefficients $\partial x^i/\partial \bar{x}^j$ as usual. But the equations here are $D_\mu V^i_{(j)} = 0$, and they are not compatible unless the alternations

$$(D_\mu D_\nu - D_\nu D_\mu) V^i_{(j)} = 0$$

are also satisfied. This leads to the condition (5.1); the tensor there must vanish identically because, when contracted with each of n *independent* contravariant vectors, the contraction must vanish; the condition is sufficient, as the vanishing of this tensor leads to the existence of solutions for the set of vectors. Finally, the vanishing of the tensor under consideration must be for the whole manifold, because the theorem is for the general path and the manifold is that explored by the totality of paths. Seen directly, if $\gamma^i_{\alpha j}$ vanishes for any path, the same is true of $\partial_\nu \gamma^i_{\alpha j}$ and therefore of the tensor in (5.1). For $m = 1$, the tensor vanishes identically, and the existence theorem for the n-spread of vectors is trivial. Any determination of $\gamma^i_{\alpha j}$ satisfying (5.1) would obviously be very useful for higher orders of partial differential equation.

One other method of approach may be noted: that is the factorization of the operator D_α by

$$D_\alpha T^{i\sigma} = T^{i\sigma}_{|\alpha} + p^k_\alpha T^{i\sigma}_{|k} - A^\tau_{\alpha\rho} T^{i\sigma}|^\rho_\tau + T^{\tau\sigma}\left(\gamma^i_{\alpha\tau} - \frac{1}{m}\gamma^i_{\nu\tau}|^\nu_k p^k_\alpha\right), \tag{5.2}$$

where

$$T^{i\sigma}_{|\alpha} = T^{i\sigma}_{,\alpha} + p^\tau_\mu \Gamma^\mu_{\alpha\rho} T^{i\sigma}|^\rho_\tau + \Gamma^\sigma_{\alpha\rho} T^{i\rho}. \tag{5.3}$$

Alternations with the operator $T^{\cdots}_{\cdots|\alpha}$ cannot give new basic tensor invariants, but we do obtain simpler equivalent sets for some of the invariants of (4.1). The formulae are

$$T^{i\lambda}_{|\alpha|\beta} - T^{i\lambda}_{|\beta|\alpha} = T^{i\rho} F^\lambda_{\rho\alpha\beta} + T^{i\lambda}|^\rho_\tau p^\tau_\mu F^\mu_{\rho\alpha\beta}, \tag{5.4}$$

where

$$F^\lambda_{\rho\alpha\beta} = \Gamma^\lambda_{\rho\alpha,\beta} - \Gamma^\lambda_{\rho\beta,\alpha} + \Gamma^\sigma_{\alpha\rho}\Gamma^\lambda_{\sigma\beta} - \Gamma^\sigma_{\beta\rho}\Gamma^\lambda_{\sigma\alpha} + p^s_\nu(\Gamma^\lambda_{\alpha\rho}|^\sigma_s \Gamma^\nu_{\beta\sigma} - \Gamma^\lambda_{\beta\rho}|^\sigma_s \Gamma^\nu_{\alpha\sigma}), \tag{5.5}$$

$$(T^{i\lambda}_{|\alpha})|^\mu_j - T^{i\lambda}|^\mu_{j|\alpha} = T^{i\rho}\Gamma^\lambda_{\alpha\rho}|^\mu_j + p^\tau_\sigma \Gamma^\sigma_{\alpha\rho}|^\mu_j T^{i\lambda}|^\rho_\tau, \tag{5.6}$$

$$T^{i\lambda}_{|\alpha|j} - T^{i\lambda}_{|j|\alpha} = T^{i\rho}\Gamma^\lambda_{\alpha\rho|j} - \frac{1}{m} T^{\tau\lambda}\gamma^i_{\nu\tau}|^\nu_{j|\alpha} + T^{i\lambda}|^\rho_\tau\{p^\tau_\sigma \Gamma^\sigma_{\alpha\rho|j} + \gamma^\tau_{\rho j|\alpha}\}. \tag{5.7}$$

This gives a symmetry between the t-space and the x-space and leads to a certain amount of simplification, particularly in cases like $n = 1$, $m > 1$.

6. Using the Grassmann–Cartan algebra, we can make a somewhat more 'geometric' approach (**3**), illustrated here for Riemann spaces. Let $e_1, e_2, ..., e_n$ be n hypercomplex units and $\alpha^i, \beta^i, ...$ ideal vectors to be identified later with differentials. For hypercomplex multiplication, we take units of higher order as

$$e^{ij} = -e^{ji} = e_i e'_j = -e'_j e_i, \quad \text{etc.}$$

In norming, we put $e_i e_j = g_{ij}$, *provided that the units come from the same ideal form and* $e_i e'_j = 0$ *for different forms* (**e**), (**e**′),

For one-dimensional exploration, the form is $\alpha^r e_r$, and the square of its norm would be a one-dimensional (squared) measure

$$ds^2 = \alpha^i \alpha^j e_i e_j = g_{ij} \dot{x}^i \dot{x}^j \, dt^2.$$

For two dimensions, we have two distinct forms $\alpha^i e_i$, $\beta^j e'_j$ to be multiplied, giving us $(\alpha^i \beta^j - \alpha^j \beta^i)(e_i e'_j - e_j e'_i)$ where the summation may be taken over all indices i, j with the sum divided by 2, or the convention made that each pair (ij) is to occur only once in the sum to avoid such factors. The norm is then the square of the differential surface measure

$$d\sigma_2^2 = (g_{ik} g_{jl} - g_{il} g_{jk}) \left(\frac{\partial x^i}{\partial t^1} \frac{\partial x^j}{\partial t^2} - \frac{\partial x^i}{\partial t^2} \frac{\partial x^j}{\partial t^1} \right) \left(\frac{\partial x^k}{\partial t^1} \frac{\partial x^l}{\partial t^2} - \frac{\partial x^k}{\partial t^2} \frac{\partial x^l}{\partial t^1} \right) dt_1^2 \, dt_2^2.$$

Here, the identifications $\alpha^i \sim x_1^i \, dt^1$, $\beta^j = x_2^j \, dt^2$, etc., are quite obvious, while the coefficient of the form is best calculated by applying our norming rules to the determinant product

$$\begin{vmatrix} e_i & e_j \\ e'_i & e'_j \end{vmatrix} \times \begin{vmatrix} e_k & e_l \\ e'_k & e'_l \end{vmatrix}. \tag{6.1}$$

The procedure is entirely similar for higher orders, the multivectors

$$\frac{\partial(x^i, x^j, ...)}{\partial(t^1, t^2, ...)}$$

being Jacobians while the coefficients are the corresponding k-minors in the matrix $||g_{ij}||$ for $1 \leqslant k \leqslant n$. For $k = n$, we get just one term in the quadratic form

$$dV_n^2 = |g_{ij}| \left| \frac{\partial x^i}{\partial t^j} \right|^2 (dt^1 ... dt^n)^2.$$

The succession of quadratic forms for $1 \leqslant k \leqslant n-1$ should enable

us to explore the Riemannian manifold by sub-sections of dimension k. For $k = 1$, we have the usual treatment of Riemann metrics. The case $k = n-1$ is also fairly easy, for the $(n-1)$-section is uniquely defined by a single vector, which is its normal in n-space, so that the treatment reduces to considerations of a supporting element defined by a point with an orienting n-vector. This leads essentially to Cartan's spaces founded on the notion of *area* instead of distance. Any k-section is conjugate to the $(n-k)$-section, so that treatment of one makes possible that of the other. The proof that the k-form is equal to $|g_{rs}p^r_\alpha p^s_\beta|$ is almost trivial. The discriminant of the k-form in the $\binom{n}{k}$ multivector Jacobians is $|g_{ij}|^{\binom{n-1}{k-1}}$, the brackets denoting binomial coefficients. The multivectors themselves have a linear transformation law with coefficients compounded from $\partial\bar{x}^i/\partial x^j$.

Our path-equations do not lend themselves immediately to such treatment, nor can such types be recovered directly (except for $m = 1$) from an inverse variational problem. The suitable generalization is not in the direction of *minimal* but of *totally geodesic* varieties. For a Riemann space, the vector p^i_α is parallel-displaced in the direction of an absolute parameter t^β if and only if

$$\frac{\partial p^i_\alpha}{\partial t^\beta} + \Gamma^i_{jk} p^j_\alpha p^k_\beta = 0; \tag{6.2}$$

but the condition is not invariant for t-transformations. Making use of the induced connexion in the parameter-space with metric

$$g_{\alpha\beta}\, dt^\alpha dt^\beta = g_{ij} p^i_\alpha p^j_\beta\, dt^\alpha dt^\beta$$

we get the usual invariant form of the condition

$$p^i_{\alpha,\beta} - \Gamma^\rho_{\alpha\beta} p^i_\rho + \Gamma^i_{jk} p^j_\alpha p^k_\beta = 0. \tag{6.3}$$

The two sets of equations coincide if and only if $\Gamma^\rho_{\alpha\beta}$ vanishes identically, the t-variety being intrinsically Euclidean. In fact, to calculate $\Gamma^\rho_{\alpha\beta}$ we must begin with

$$\begin{aligned} g_{\alpha\beta,\gamma} = (g_{ij} p^i_\alpha p^j_\beta)_{,\gamma} &= g_{ij,k} p^i_\alpha p^j_\beta p^k_\gamma + g_{ij} p^i_{\alpha,\gamma} p^j_\beta + g_{ij} p^i_\alpha p^j_{\beta,\gamma} \\ &= g_{ir} p^r_\beta \{p^i_{\alpha,\gamma} + \Gamma^i_{ks} p^k_\alpha p^s_\gamma\} + g_{ir} p^r_\alpha \{p^i_{\beta,\gamma} + \Gamma^i_{ks} p^k_\beta P^s_\gamma\}. \end{aligned} \tag{6.4}$$

In this case the tensor $A^{iv}_{j\alpha\beta}$ vanishes identically; we can identify $H^i_{\alpha\beta}$ with $\Gamma^i_{jk} p^j_\alpha p^k_\beta$ and so consider the path-space as a Riemannian

manifold explored by means of totally geodesic sub-spaces. Keeping in mind the result that

$$|g_{ij}p^i_\alpha p^j_\beta| \neq 0 \quad \text{if} \quad |g_{ij}| \neq 0,$$

we see from the constancy of the integrals $g_{ij}p^i_\alpha p^j_\beta$ that $t^1, t^2, \ldots$ are all arc-like parameters and that we can take the t-space as flat. For $m = 1$, the one-dimensional geodesic variety has the parameter metric

$$(g_{ij}\dot{x}^i\dot{x}^j)\,dt^2 = k\,dt^2,$$

making use of the fact that $g_{ij}\dot{x}^i\dot{x}^j$ is constant along the paths.

The proper inverse question to ask, therefore, seems to be: 'What are the manifolds for which $\partial^2 x^i/\partial t^\alpha \partial t^\beta + H^i_{\alpha\beta} = 0$ represent totally geodesic varieties; and in what sense?'

7. As we have seen from the preceding sections, the inverse variational problem for partial differential systems has not the same value as for ordinary differential equations. However, the formal apparatus is still of some interest in 'unfolding' the identities that coalesce for a single absolute parameter with $m = 1$. The Eulerian equations are

$$\delta \int \phi(x^i, p^j_\alpha, t^\beta)\,dt^1 \ldots dt^m = 0 \to \delta_i\phi \equiv \partial_\alpha \phi|^\alpha_i - \phi_{,i} = 0. \tag{7.1}$$

In reducing these to their invariantive form, we substitute from (1.5) to get

$$\delta_i\phi \equiv D_\alpha\phi|^\alpha_i - \Gamma^\nu_{\nu\rho}\phi|^\rho_i + p\Gamma^\alpha_{\alpha\rho}\phi|^\rho_i + q\gamma^r_{\alpha r}\phi|^\alpha_i - \phi_{|i}. \tag{7.2}$$

This gives us at once the result that *the Euler equations* $\delta_i\phi = 0$ *are tensorial if and only if* ϕ *is of x-weight zero and t-weight unity*, i.e. $p = 1$, $q = 0$. This condition may therefore be assumed hereafter, though both the direct and the inverse problems can be handled without it simply by abandoning tensor-invariance. The proper invariantive form is

$$\delta_i\phi \equiv D_\alpha\phi|^\alpha_i - \phi_{|i} = 0. \tag{7.2a}$$

We shall, moreover, need the identities:

$$\left.\begin{aligned} \delta_i\phi &\equiv \phi|^\rho_{i,\rho} + \phi|^\rho_{i,r}p^r_\rho - \phi|^\rho_i|^\mu_r H^r_{\rho\mu} - \phi_{,i} \\ (\delta_i\phi)|^\alpha_j &\equiv -\phi|^\rho_i|^\mu_r H^r_{\rho\mu}|^\alpha_j + \partial_\rho(\phi|^\rho_i|^\alpha_j) + (\phi|^\alpha_{i,j} - \phi|^\alpha_{j,i}) \\ &\equiv D_\rho(\phi|^\rho_i|^\alpha_j) + (\phi|^\alpha_{i|j} - \phi|^\alpha_{j|i}) - \phi|^\rho_i|^\nu_r A^{r\alpha}_{j\nu\rho} + \phi|^\alpha_r T^r_{ij} \end{aligned}\right\}. \tag{7.2b}$$

The questions now to be settled are:

(i) when are the equations of variation of $x^i_{,\alpha,\beta} + H^i_{\alpha\beta} = 0$ self-adjoint?

(ii) are the conditions thus obtained, known to be necessary, also sufficient for derivation from a variational principle?

Once again, *self-adjoincy has no meaning unless some tensor* $\phi_{ij}^{\alpha\beta}$ *is found to contract with the path-equations*; for the adjoint can only be taken to mean an equation whose solutions (if any) are integrating factors of the original, and conversely. We then vary

$$\phi_{ir}^{\alpha\beta}\left(\frac{\partial^2 x^r}{\partial t^\alpha \partial t^\beta}+H_{\alpha\beta}^i\right)=0. \tag{7.3}$$

Of course, no equivalent of the condition $|f_{;i;j}| \neq 0$ which has to be imposed for $m=1$ can be expected here as we do not mean to pass from the Euler equations to those of the paths. Giving a vector variation u^i to x^i, we may vary (7.3) directly, and then, as for $m=1$, require u^i to be an integrating factor of the varied equations. In deriving the resultant identities, we must not eliminate second and higher derivatives of the x^i in view of the paths, so that the coefficients of terms of each degree in $x^i_{,\alpha,\beta}$, $x^i_{,\alpha,\beta,\nu}$,... may be equated. The results are easily seen to be as follows:

$$\left.\begin{array}{ll}\text{(a)} & \phi_{ij}^{\alpha\beta} \text{ does not contain any derivative of } x \text{ higher} \\ & \text{than } p_\sigma^i, \\ \text{(b)} & \phi_{ij}^{\alpha\beta} \text{ must have the form } \frac{1}{2}\{\phi_i|^\alpha|_j^\beta+\phi_j|^\alpha|_i^\beta\}, \text{ as was to} \\ & \text{be expected from the Euler equations whose} \\ & \text{leading term is } \phi_i|^\alpha|_j^\beta x^j_{,\alpha,\beta}+\ldots.\end{array}\right\} \tag{7.4}$$

There follow two conditions of more complicated type

$$2\partial_\rho \phi_{ij}^{\rho\nu}-\phi_{rj}^{\alpha\beta}H^r_{\alpha\beta}|_i^\nu-\phi_{ri}^{\alpha\beta}H^r_{\alpha\beta}|_j^\nu=0, \tag{7.5}$$

$$\phi_{rj}^{\alpha\beta}H^r_{\alpha\beta,i}-\phi_{ri}^{\alpha\beta}H^r_{\alpha\beta,j}+\tfrac{1}{2}\partial_\nu\{\phi_{ri}^{\alpha\beta}H^r_{\alpha\beta}|_j^\nu-\phi_{rj}^{\alpha\beta}H^r_{\alpha\beta}|_i^\nu\}=0. \tag{7.6}$$

These can be put into an invariantive form if and only if $\phi_{ij}^{\alpha\beta}$ is of t-weight unity and x-weight zero. The first is then reduced to

$$2D_\rho \phi_{ij}^{\rho\nu}-\phi_{rj}^{\alpha\beta}A_{i\alpha\beta}^{r\nu}-\phi_{ir}^{\alpha\beta}A_{j\alpha\beta}^{r\nu}=0. \tag{7.7}$$

This condition, or (7.5), must itself be utilized in the reduction of (7.6), which then becomes, after removing the factor $\frac{1}{2}$,

$$\phi_{rj}^{\alpha\beta}\{P^r_{i\alpha\beta}+P^r_{i\beta\alpha}\}-\phi_{ri}^{\alpha\beta}\{P^r_{j\alpha\beta}+P^r_{j\beta\alpha}\}+ \\ +D_\nu\{\phi_{ri}^{\alpha\beta}A_{j\alpha\beta}^{r\nu}-\phi_{rj}^{\alpha\beta}A_{i\alpha\beta}^{r\nu}\}=0. \tag{7.8}$$

One way of avoiding excessive calculation in the reduction is to generalize the method of using special coordinates, which is not strictly available except when $m=1$, unless $P^i_{j\alpha\beta}-P^i_{j\beta\alpha}=0$. In that case, one could have chosen special coordinates which made $\gamma^i_{\alpha j}$ zero along a path, hence also $\partial_\beta \gamma^i_{\alpha j}$, etc. The reductions then

become evident, but as the special condition upon $P^i_{j\alpha\beta}$ does not hold in general, the general case implies extra terms containing the tensor assumed to vanish. In the present situation, this must appear as a linear additive term, and direct comparison determines the coefficients at once. For $m = 1$, both $A^{ir}_{j\alpha\beta}$ and $P^i_{j\alpha\beta} - P^i_{j\beta\alpha}$ vanish identically, and this makes all reductions much simpler.

The inverse problem is now seen to take on the following form: Given a function ϕ whose second p-derivatives satisfy conditions (7.4)–(7.6), can linear additive terms $P^\rho_r(x,t)p^r_\rho + Q(x,t)$ be so chosen as to make $\delta_i(\phi + P^\rho_r p^r_\rho + Q) = 0$?

Analytically we must have

$$\delta_i \phi = (P^\alpha_{r,i} - P^\alpha_{i,r})p^r_\alpha + (Q_{,i} - P^\rho_{i,\rho}). \tag{7.9}$$

In this equation, we have to remember that ϕ is known, and that the known Eulerian derivative $\delta_i \phi$ must be shown, in view of the identities (7.4)–(7.6), to have the structure in equations (7.9). Now the bracketed terms on the right being independent of $p^{\dot{i}}_\alpha$, we note that $(\delta_i \phi)|^\alpha_j$ must have the form $P^\alpha_{j,i} - P^\alpha_{i,j}$, which leads to three conditions:

$$\left.\begin{aligned}
&[\mathrm{A}] \quad (\delta_i\phi)|^\alpha_j + (\delta_j\phi)|^\alpha_i = 0; \quad [(\delta_i\phi)|^\alpha_j \text{ antisymmetric in } i,j],\\
&[\mathrm{A}'] \quad (\delta_i\phi)|^\alpha_j|^\beta_k = 0; \quad [(\delta_i\phi) \text{ linear in } p^i_\nu)],\\
&[\mathrm{B}'] \quad [\text{from } (\delta_i\phi)|^\alpha_j, \text{ an } x\text{-curl}]\\
&\qquad\qquad (\delta_i\phi)|^\alpha_{j,k} + (\delta_j\phi)|^\alpha_{k,i} + (\delta_k\phi)|^\alpha_{i,j} = 0,
\end{aligned}\right\} \tag{7.10}$$

i.e.

$$(\delta_i\phi)|^\alpha_{j|k} + (\delta_j\phi)|^\alpha_{k|i} + (\delta_k\phi)|^\alpha_{i|j} - \\ -\{(\delta_i\phi)|^\alpha_r T^r_{jk} + (\delta_j\phi)|^\alpha_r T^r_{ki} + (\delta_k\phi)|^\alpha_r T^r_{ij}\} = 0.$$

Finally, we obtain the condition of integrability $Q_{,i,j} - Q_{,j,i} = 0$ as

$$[\mathrm{B}] \quad (\delta_i\phi)_{,j} - (\delta_j\phi)_{,i} - (\delta_i\phi)|^\rho_{j,\rho} - (\delta_i\phi)|^\rho_{j,r} p^r_\rho = 0, \tag{7.10}$$

i.e.
$$(\delta_i\phi)_{|j} - (\delta_j\phi)_{|i} - D_\rho\{(\delta_i\phi)|^\rho_j\} - T^r_{ij}\delta_r\phi = 0.$$

The reduction of (7.10) [A] and [A′] is direct, if we keep in mind the given result that

$$\phi^{\alpha\beta}_{ij} = \tfrac{1}{2}\{\phi|^\alpha_i|^\beta_j + \phi|^\alpha_j|^\beta_i\}.$$

For
$$(\delta_i\phi)|^\nu_j + (\delta_j\phi)|^\nu_i \equiv 2D_\rho\phi^{\rho\nu}_{ij} - \phi^{\alpha\beta}_{ir}A^{r\nu}_{j\alpha\beta} - \phi^{\alpha\beta}_{rj}A^{r\nu}_{i\alpha\beta} = 0. \tag{7.11}$$

$$(\delta_i\phi)|^\nu_j|^\mu_k \equiv \{D_\rho\phi^{\rho\nu}_{ik} - \tfrac{1}{2}A^{r\nu}_{k\alpha\beta}\phi^{\alpha\beta}_{ir} - \tfrac{1}{2}A^{r\nu}_{i\alpha\beta}\phi^{\alpha\beta}_{rk}\}|^\mu_j + \ldots + \ldots$$

(two other sets of terms by cyclic rotation of i, j, k)

$$= 0. \tag{7.12}$$

Hence, *the condition of linearity is a consequence of that of anti-symmetry* (as for ordinary differential equations).

For the proof of (7.10) [B] we use the result

$$(\delta_i\phi)_{,j}-(\delta_j\phi)_{,i}-(\delta_i\phi)|^{\mu}_{j,\mu}-(\delta_i\phi)|^{\mu}_{j,r}p^r_{\mu}$$
$$\equiv \phi^{\rho\mu}_{jr}H^r_{\rho\mu,i}-\phi^{\rho\mu}_{ir}H^r_{\rho\mu,j}+\partial_\alpha(\phi^{\rho\mu}_{ir}H^r_{\rho\mu}|^{\alpha}_{j})-$$
$$-\partial_\alpha\partial_\rho\phi^{\rho\alpha}_{ij}-(\delta_i\phi)|^{\alpha}_{r}|^{\mu}H^r_{\alpha\mu}. \quad (7.13)$$

For the particular function ϕ, we know further that the final term on the right vanishes, while the one just preceding, in view of (7.5), is merely

$$-\tfrac{1}{2}\partial_\rho\{\phi^{\alpha\beta}_{rj}H^r_{\alpha\beta}|^{\rho}_{i}+\phi^{\alpha\beta}_{ri}H^r_{\alpha\beta}|^{\rho}_{j}\}.$$

Thus we get

$$(\delta_i\phi)_{,j}-(\delta_j\phi)_{,i}-(\delta_i\phi)|^{\rho}_{j,\rho}-(\delta_i\phi)^{\rho}_{j,r}p^r_{\rho}$$
$$\equiv \phi^{\rho\mu}_{jr}H^r_{\rho\mu,i}-\phi^{\rho\mu}_{ir}H^r_{\rho\mu,j}+\tfrac{1}{2}\partial_\alpha\{\phi^{\rho\mu}_{ir}H^r_{\rho\mu}|^{\alpha}_{j}-\phi^{\rho\mu}_{ij}H^r_{\rho\mu}|^{\alpha}_{i}\}-$$
$$-\tfrac{1}{2}\partial_\nu\{2\partial_\rho\phi^{\rho\nu}_{ij}-\phi^{\alpha\beta}_{rj}H^r_{\alpha\beta}|^{\nu}_{i}-\phi^{\alpha\beta}_{ri}H^r_{\alpha\beta}|^{\nu}_{j}\}-(\delta_i\phi)|^{\alpha}_{j}|^{\mu}_{r}H^r_{\alpha\mu}. \quad (7.14)$$

This vanishes in terms of (7.5) and (7.6), which proves (7.10) [B].

For the remaining equations (7.10) [B′], i.e. to prove that $(\delta_i\phi)|^{\alpha}_{j}$ is an x-curl, I save space and make the structure of the proof more transparent by making use of known properties of the $\delta_i\phi$ in question. We know from (7.2 b) that

$$(\delta_i\phi)|^{\alpha}_{j} \equiv -\phi^{\rho\mu}_{ir}H^r_{\rho\mu}|^{\alpha}_{j}+\partial_\rho(\phi|^{\rho}_{i}|^{\alpha}_{j})+(\phi|^{\alpha}_{i,j}-\phi|^{\alpha}_{j,i}),$$

of which the last term is already an x-curl. The principal difficulty is occasioned by the terms $\partial_\rho(\phi|^{\rho}_{i}|^{\alpha}_{j})$; our basic conditions (7.4)–(7.6) deal only with the symmetric part of this. The necessary symmetrization can be effected by operating once more with ∂_α upon the whole of the above identity. Then we may write

$$\partial_\alpha(\delta_i\phi)|^{\alpha}_{j} \equiv -\partial_\nu\{\phi^{\rho\mu}_{ir}H^r_{\rho\mu}|^{\nu}_{j}\}+\partial_\rho\partial_\nu\phi^{\rho\nu}_{ij}+H^r_{\rho\mu,j}\phi^{\rho\mu}_{ir}-$$
$$-H^r_{\rho\mu,i}\phi^{\rho\mu}_{jr}+(\partial_\nu\phi|^{\nu}_{i})_{,j}-(\partial_\nu\phi|^{\nu}_{j})_{,i}. \quad (7.15)$$

The second term on the right being eliminated as before by differentiation of (7.5), we get, *for the particular* ϕ,

$$\partial_\alpha\{(\delta_i\phi)|^{\alpha}_{j}\} = \tfrac{1}{2}\partial_\nu\{\phi^{\rho\mu}_{rj}H^r_{\rho\mu}|^{\nu}_{i}-\phi^{\rho\mu}_{ri}H^r_{\rho\mu}|^{\nu}_{j}\}+\phi^{\rho\mu}_{ir}H^r_{\rho\mu,j}-$$
$$-\phi^{\rho\mu}_{jr}H^r_{\rho\mu,i}+(\partial_\nu\phi|^{\nu}_{i})_{,j}-(\partial_\nu\phi|^{\nu}_{j})_{,i}. \quad (7.16)$$

This shows, even without calculating the rest of (7.10) [B′] that *the divergence* $\partial_\alpha(\delta_i\phi)|^{\alpha}_{j}$ *is an* x*-curl* since ϕ satisfies (7.6). Now we know from (7.10) [A′] that $(\delta_i\phi)|^{\alpha}_{j}$ is here independent of p^k_β, so that its divergence obtained by the operation ∂_α contains a term independent

of p_β^k and another set linear therein, each of which must be an x-curl. The coefficients of the linear terms are therefore merely x-derivatives of an x-curl. All the additive terms left undetermined by this process do not matter, for the integrand of any problem in the calculus of variations may be changed by arbitrary additive terms which constitute a perfect differential. This completes the rather clumsy proof.

8. The special case $n = 1$ may be considered briefly because it possesses a complete differential geometry of its own, though no interpretation is possible as 'exploration' by means of sub-spaces. The path-equations are

$$\frac{\partial^2 x}{\partial t^\alpha \partial t^\beta} + H_{\alpha\beta}(x, t^\mu, p_\nu) = 0, \qquad p_\alpha = \frac{\partial x}{\partial t^\alpha} \quad (\alpha, \beta = 1, 2, \ldots, m). \tag{8.1}$$

The supporting element is here defined, for a single function x spread over the t-space, by the covariant vector p_α at a point $(t^1, \ldots, t^m)$. We may specialize all the formulae of the preceding sections, but it is possible to make important modifications as follows. The tensor $A^{i\nu}_{j\alpha\beta}$ now becomes

$$A^\nu_{\alpha\beta} = \Gamma^\nu_{\alpha\beta} - \delta^\nu_\beta \gamma_\alpha - \delta^\nu_\alpha \gamma_\beta + H_{\alpha\beta}|^\nu. \tag{8.2}$$

Thus $A^\nu_{\alpha\beta} = 0$ is no longer unnecessarily restrictive, and, in fact, this would enable us to obtain a simple three-index connexion analogous to that given by projective transformations of a symmetric affine connexion. Suppressing x-transformations altogether, since they are of no essential importance to the t-space, we may set $\gamma_\alpha = 0$, thus getting

$$\Gamma^\nu_{\alpha\beta} = -H_{\alpha\beta}|^\nu = -\frac{\partial H_{\alpha\beta}}{\partial p_\nu}. \tag{8.3}$$

Of the fundamental invariants, the following three lose their importance in the absence of x-transformations.

$$\left.\begin{aligned} &\text{(a)} \quad P_{\alpha\beta} = \frac{\partial H_{\alpha\beta}}{\partial x} - H_{\alpha\beta}|^\sigma \gamma_\sigma + \gamma_\beta \gamma_\alpha - \gamma_{\alpha,\beta} - p_\beta \frac{\partial \gamma_\alpha}{\partial x} + H_{\rho\beta} \gamma_\alpha |^\rho \\ &\text{(b)} \quad C_\alpha = \frac{\partial \gamma_\alpha}{\partial x} - \gamma_\beta \gamma_\alpha |^\beta - \frac{1}{m} \{\partial_\alpha \gamma_\beta |^\beta - \gamma_\alpha \gamma_\beta |^\beta\} \\ &\text{(c)} \quad \Gamma^\nu_{\alpha\beta|j} \to \frac{\partial \Gamma^\nu_{\alpha\beta}}{\partial x} - \gamma_\rho \Gamma^\nu_{\alpha\beta}|^\rho \end{aligned}\right\}. \tag{8.4}$$

Of the remainder, $H^i_{\alpha\beta}|^\nu_j|^\mu_k|^\rho_l$ survives, but can actually be replaced by a derivative of the second order if we use the special determination in (8.3) of the connexion coefficients. We then get $H_{\alpha\beta}|^\mu|^\nu$ as a differential invariant of lower order (suppressing x-transformations). Finally, we have the generalized Riemann–Christoffel curvature tensor $R^\nu_{\sigma\alpha\beta}$.

For the general formulae, it is necessary to keep in mind the silent x-index. Besides $\partial/\partial p_\alpha$ and $\partial/\partial x$, we have the operator of covariant differentiation from (5.3)

$$T^{\cdots}_{\cdots|\alpha} = T^{\cdots}_{\cdots,\alpha} + p_\mu \Gamma^\mu_{\alpha\rho} T^{\cdots}_{\cdots}|^\rho + \Gamma^\sigma_{\alpha\rho} T^{\cdots}_{\sigma\cdots} + \ldots - \Gamma^\rho_{\alpha\mu} T^{\cdots}_{\rho\cdots} - \ldots \quad (8.5)$$

with which D_α may be replaced as indicated in (5.2).

The inverse variational problem may be regarded as that of determining when a generalized Laplace equation can be associated with the paths (8.1), though the paths cannot be recovered from a single equation of the sort except by an indirect process. The equations (7.4) say that a symmetric contravariant second-rank $g^{\alpha\beta}$ must exist essentially as the second derivative $\phi|^\alpha|^\beta$ of some $\phi(x, t^\alpha, p_\mu)$ of weight unity. Then (7.5) says that the *contracted* covariant derivative $D_\rho g^{\rho\nu}$ must vanish, where we must keep in mind the fact that one term $-g^{\rho\nu}\Gamma^\mu_{\rho\mu}$ must represent the unit weight of the tensor. The condition (7.8) is seen to hold by vanishing identically.

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THE MAXIMUM TERM OF AN ENTIRE SERIES (III)

By S. M. SHAH (*Aligarh*)

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1. LET $f(z) = \sum_0^\infty a_n z^n$ be an entire function of order ρ $(0 \leqslant \rho \leqslant \infty)$ and lower order λ and let $\mu(r)$ denote the maximum term of the series for $|z| = r$, $\nu(r)$ the rank of this term. It is known that [**1, 2, 3**]

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log \mu(r)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log \nu(r)}{\log r}, \tag{1}$$

$$\lambda = \underline{\lim_{r\to\infty}} \frac{\log\log \mu(r)}{\log r} = \underline{\lim_{r\to\infty}} \frac{\log \nu(r)}{\log r}, \tag{2}$$

$$\underline{\lim_{r\to\infty}} \frac{\log \mu(r)}{\nu(r)} \leqslant \frac{1}{\rho} \leqslant \frac{1}{\lambda} \leqslant \overline{\lim_{r\to\infty}} \frac{\log \mu(r)}{\nu(r)}. \tag{3}$$

If $0 < \rho < \infty$, then $\log M(r) \sim \log \mu(r)$ and [**3**]

$$\underline{\lim_{r\to\infty}} \frac{\nu(r)}{r^\rho} \leqslant \rho \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^\rho} \leqslant \overline{\lim_{r\to\infty}} \frac{\nu(r)}{r^\rho}. \tag{4}$$

Let

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^\rho} = \begin{cases} T \\ t \end{cases}, \qquad \overline{\lim_{r\to\infty}} \frac{\nu(r)}{r^\rho} = \begin{cases} \gamma \\ \delta \end{cases}, \qquad \overline{\lim_{r\to\infty}} \frac{\log \mu(r)}{\nu(r)} = \begin{cases} c \\ d \end{cases}.$$

I prove here

THEOREM 1. *If* $f(z) = \sum_0^\infty a_n z^n$ *is an entire function of order* ρ $(0 < \rho < \infty)$, *then*

$$\delta \leqslant \frac{\gamma}{e} e^{\delta/\gamma} \leqslant \rho T \leqslant \gamma, \tag{5}$$

$$\delta \leqslant \rho t \leqslant \delta\left(1 + \log\frac{\gamma}{\delta}\right) \leqslant \gamma. \tag{6}$$

From (5) and (6) it follows that, if $\nu(r) \sim \gamma r^\rho$, then $\log M(r) \sim \frac{\gamma}{\rho} r^\rho$ and $c = d = 1/\rho$.

The converse that, if $\log M(r) \sim T r^\rho$, then $\nu(r) \sim \rho T r^\rho$ is also true [**4**, 11].

We also have, from (5) and (6),

$$\rho T \leqslant \gamma \leqslant e\rho T, \qquad \delta \leqslant \rho t.$$

If we take $f(z) = \exp(kz)$ $(k > 0)$, we see that the inequalities $\rho T \leqslant \gamma$ and $\delta \leqslant \rho t$ are best possible. I show that

$$\gamma \leqslant e\rho T, \qquad \rho t \leqslant \delta\left(1+\log\frac{\gamma}{\delta}\right)$$

are best possible.

THEOREM 2. (i) *For each positive ρ and positive T there exists an entire function $f(z)$ of order ρ and type T such that*

$$\gamma = \overline{\lim_{r\to\infty}}\frac{\nu(r)}{r^\rho} = \rho e T.$$

(ii) *For each positive ρ and positive δ and $\gamma > \delta$, there exists an entire function $f(z)$ of order ρ such that*

$$\overline{\lim_{r\to\infty}}\frac{\nu(r)}{r^\rho} = \begin{cases}\gamma\\ \delta\end{cases}$$

and

$$t = \lim_{r\to\infty}\frac{\log M(r)}{r^\rho} = \frac{\delta}{\rho}\left(1+\log\frac{\gamma}{\delta}\right).$$

2. Proof of Theorem 1

$$\begin{aligned}\log M(rk^{1/\rho}) &\sim \log\mu(rk^{1/\rho})\\ &= k_1+\left(\int_{r_0}^{r}+\int_{r}^{rk^{1/\rho}}\frac{\nu(x)}{x}\,dx\right)\\ &> k_2+\frac{(\delta-\epsilon)r^\rho}{\rho}+\frac{\nu(r)}{\rho}\log k.\end{aligned}$$

Hence

$$k\frac{\log M(rk^{1/\rho})}{kr^\rho} > o(r^\rho)+\frac{\delta-\epsilon}{\rho}+\frac{\nu(r)}{r^\rho}\frac{\log k}{\rho},$$

and so

$$kt \geqslant \frac{\delta+\delta\log k}{\rho}, \qquad kT \geqslant \frac{\delta+\gamma\log k}{\rho}.$$

Taking $k = 1$ in the first inequality and $\log k = (\gamma-\delta)/\gamma$ in the second, we get

$$t \geqslant \frac{\delta}{\rho}, \qquad \rho T e \geqslant \gamma e^{\delta/\gamma}.$$

Since, for $x \geqslant 0$, $(\exp x)/x \geqslant e$ we have $\gamma\exp(\delta/\gamma) \geqslant e\delta$. Further

$$\log M(rk^{1/\rho}) \leqslant k_3+\frac{(\gamma+\epsilon)r^\rho}{\rho}+\frac{\nu(rk^{1/\rho})\log k}{\rho}.$$

Hence

$$kT \leqslant \frac{\gamma+\gamma k\log k}{\rho}, \qquad kt \leqslant \frac{\gamma+\delta k\log k}{\rho}.$$

Taking $k = 1$ in first inequality and $k = \gamma/\delta$ in the second, we get

$$T \leqslant \gamma/\rho, \qquad \rho t \leqslant \delta\left(1+\log\frac{\gamma}{\delta}\right) \leqslant \delta\frac{\gamma}{\delta} = \gamma.$$

3. Proof of Theorem 2 (i)

Let

$$\lambda_1 = 1, \qquad P = 1+[\rho], \qquad \lambda_{n+1} = \exp(\lambda_n^P) \quad (n = 1, 2, \ldots),$$

and

$$f(z) = \sum_1^\infty \left(\frac{z}{\lambda_n}\right)^{[e\rho T\lambda_n^\rho]},$$

which is an entire function of order ρ. Further

$$(r/\lambda_n)^{[e\rho T\lambda_n^\rho]}$$

is the maximum term if $\phi(n) \leqslant r < \phi(n+1)$, where

$$\phi(n) = \{\lambda_n^{[e\rho T\lambda_n^\rho]}/\lambda_{n-1}^{[e\rho T\lambda_{n-1}^\rho]}\}^{1/\{[e\rho T\lambda_n^\rho]-[e\rho T\lambda_{n-1}^\rho]\}}.$$

Let k be a constant such that $\max(0, 1-\rho) < k < 1$. Then, for $n > n_0$,

$$\lambda_n < \phi(n) < \lambda_n+\lambda_n^k < \lambda_n e^{1/\rho} < \lambda_{n+1} < \phi(n+1),$$

and hence, if $\phi(n) \leqslant r < \phi(n+1)$,

$$\frac{\log M(r)}{r^\rho} \sim \frac{\log \mu(r)}{r^\rho} \sim \frac{e\rho T\lambda_n^\rho \log(r/\lambda_n)}{r^\rho} = eT\frac{\log(r/\lambda_n)^\rho}{(r/\lambda_n)^\rho} \leqslant T,$$

since $\log x/x \leqslant 1/e$. Further, if $R(n) = \lambda_n e^{1/\rho}$,

$$\log M\{R(n)\}/\{R(n)\}^\rho \sim eT/e = T.$$

Hence

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^\rho} = T.$$

Further

$$\frac{\log M(\lambda_n)}{\lambda_n^\rho} \sim \frac{e\rho T\lambda_{n-1}^\rho \log(\lambda_n/\lambda_{n-1})}{\lambda_n^\rho} \to 0,$$

as $n \to \infty$. Hence $t = 0$, and so $\delta = 0$. Let now

$$r = \lambda_n+\lambda_n^k.$$

Then

$$\frac{\nu(r)}{r^\rho} \sim \frac{e\rho T\lambda_n^\rho}{(\lambda_n+\lambda_n^k)^\rho} \sim e\rho T,$$

and hence

$$\gamma = e\rho T.$$

4. Proof of Theorem 2 (ii)

Let

$$n_1 = 2, \qquad n_{s+1} = n_s^4 \quad (s = 1, 2, \ldots), \qquad r_1 = 1,$$

$$r_m = (m/\gamma)^{1/\rho} \quad \text{for} \quad n_s \leqslant m \leqslant (n_{s+1})\delta/\gamma;$$

$$r_m = \left(\frac{n_{s+1}}{\gamma}\right)^{1/\rho} - \frac{n_{s+1}-m}{(n_{s+1})!} \quad \text{for} \quad (n_s+1)\delta/\gamma < m \leqslant n_{s+1} \quad (s = 1, 2, \ldots),$$

and let $$f(z) = 1 + \sum_{1}^{\infty} \frac{z^n}{r_1 r_2 \ldots r_n}.$$

Then $f(z)$ is an entire function of order ρ. Further

$$\varliminf_{r\to\infty} \frac{\nu(r)}{r^\rho} = \varliminf_{m\to\infty} \frac{m}{r_{m+1}^\rho} = \lim_{s\to\infty} \frac{(n_{s+1})\delta/\gamma}{(n_{s+1})/\gamma} = \delta,$$

$$\varlimsup_{r\to\infty} \frac{\nu(r)}{r^\rho} = \varlimsup_{m\to\infty} \frac{m}{r_m^\rho} = \lim_{m\to\infty} \frac{m}{m/\gamma} = \gamma.$$

Further $$t = \varliminf_{r\to\infty} \frac{\log M(r)}{r^\rho} = \varliminf_{r\to\infty} \frac{\log \mu(r)}{r^\rho}.$$

If $r_n \leqslant r < r_{n+1}$,

$$\frac{\log \mu(r)}{r^\rho} = \frac{1}{r^\rho} \log\left(\frac{r^n}{r_1 r_2 \ldots r_n}\right). \tag{7}$$

Let $r = (r_{n+1} - a)$ where

$$0 < a < \min\left(\frac{r_{n+1} - r_n}{2}, 1\right) \quad \text{and} \quad n = \left[\frac{\delta n_{s+1}}{\gamma}\right].$$

The expression on the right of (7)

$$= \frac{1}{(r_{n+1}-a)^\rho}\{n \log(r_{n+1}-a) - \log(r_1 r_2 \ldots r_n)\}$$

$$\sim \frac{\gamma}{n_{s+1}}\left\{\frac{(n_{s+1})\delta}{\gamma} \log\left(\frac{n_{s+1}}{\gamma}\right)^{1/\rho} - \frac{(n_{s+1})\delta}{\gamma\rho} \log \frac{\delta n_{s+1}}{\gamma} + \frac{\delta}{\gamma\rho} n_{s+1} + \frac{(n_{s+1})\delta \log \gamma}{\gamma\rho}\right\}$$

$$\to \frac{\delta}{\rho}\left(1 + \log \frac{\gamma}{\delta}\right) \quad \text{as } s \to \infty.$$

It is easily seen that

$$t = \frac{\delta}{\rho}\left(1 + \log \frac{\gamma}{\delta}\right), \qquad T = \varlimsup_{r\to\infty} \frac{\log M(r)}{r^\rho} = \frac{\gamma}{\rho}.$$

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BOUNDED INTEGRAL TRANSFORMS

By I. J. GOOD and G. E. H. REUTER (*Manchester*)

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1. Notation

We shall consider real-valued functions defined over the interval $(0, \infty)$. If f^2 is Lebesgue-integrable over $(0, \infty)$, we shall write $f \in L^2$; functions which differ only on a null set will be regarded as identical.

The phrase 'Mellin transform' will be used in the following sense. Let $s = \frac{1}{2} + i\tau$ $(-\infty < \tau < \infty)$; if $\int_{-\infty}^{\infty} |\mathfrak{F}(s)|^2 \, d\tau < \infty$, we say that $\mathfrak{F} \in \mathfrak{L}^2$. Then† there is a $(1, 1)$ correspondence between functions $f \in L^2$ and functions $\mathfrak{F} \in \mathfrak{L}^2$ such that $\mathfrak{F}(\bar{s}) = \overline{\mathfrak{F}(s)}$, given by

$$\mathfrak{F}(s) = \underset{X \to \infty}{\text{l.i.m.}} \int_{X^{-1}}^{X} x^{s-1} f(x) \, dx, \qquad f(x) = \underset{T \to \infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-T}^{T} \mathfrak{F}(s) x^{-s} \, d\tau.$$

Moreover,

$$\int_0^{\infty} f^2(x) \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathfrak{F}(s)|^2 \, d\tau \quad \text{(Parseval formula).}$$

Under these circumstances, we say that $f(x)$ has the Mellin transform $\mathfrak{F}(s)$.

2. Introduction

The following generalization of Plancherel's theorem for Fourier transforms is due‡ essentially to Watson (**11**); in Watson's original statement the Parseval formula was part of the hypotheses, whilst the statement given below is due to Plancherel (**5**).

THEOREM A. *Suppose that to every $f \in L^2$ there corresponds a $g \in L^2$ which satisfies (for all $x \geqslant 0$)*

$$\int_0^x g(t) \, dt = \int_0^{\infty} f(t) \frac{K(xt)}{t} \, dt, \qquad \int_0^x f(t) \, dt = \int_0^{\infty} g(t) \frac{K(xt)}{t} \, dt.$$

For this, it is necessary and sufficient that, for all $a \geqslant 0$, $b \geqslant 0$,

$$\int_0^{\infty} \frac{K(ax) K(bx)}{x^2} \, dx = \min(a, b)$$

† For proofs of the statements which follow, see Titchmarsh (**10**), 94–5.

‡ See also Titchmarsh (**9**), Plancherel (**5**), and Busbridge (**2**).

or, equivalently, that $K(x)/x$ should have a Mellin transform $k(s)/(1-s)$ with $|k(s)| = 1$. Moreover,

$$\int_0^\infty g^2(x)\,dx = \int_0^\infty f^2(x)\,dx.$$

This theorem has been extended in two directions. First, there is a result found† independently by Bochner (**1**) and Kaczmarz (**3**).

THEOREM B. *Suppose that to every $f \in L^2$ there correspond g, $h \in L^2$ such that (for all $x \geqslant 0$)*

$$\int_0^x g(t)\,dt = \int_0^\infty f(t)\phi(x,t)\,dt, \qquad \int_0^x h(t)\,dt = \int_0^\infty f(t)\psi(x,t)\,dt, \quad (B_1)$$

$$\int_0^x f(t)\,dt = \int_0^\infty g(t)\psi(x,t)\,dt = \int_0^\infty h(t)\phi(x,t)\,dt, \quad (B_2)$$

$$\int_0^\infty g^2(x)\,dx = \int_0^\infty h^2(x)\,dx = \int_0^\infty f^2(x)\,dx. \quad (B_3)$$

For all this, it is necessary and sufficient that, for all $x \geqslant 0$, $y \geqslant 0$,

$$\int_0^\infty \phi(x,t)\phi(y,t)\,dt = \min(x,y) = \int_0^\infty \psi(x,t)\psi(y,t)\,dt, \quad (B_4)$$

$$\int_0^y \phi(x,t)\,dt = \int_0^x \psi(y,t)\,dt. \quad (B_5)$$

Theorem B does not include Theorem A since the Parseval formulae (B_3) are now part of the hypotheses. On the other hand, Kober (**4**) has proved the following extension of Theorem A.

THEOREM C. *In order that to every $f \in L^2$ there shall correspond g, $h \in L^2$ such that (for all $x \geqslant 0$)*

$$\int_0^x g(t)\,dt = \int_0^\infty f(t)\frac{K(xt)}{t}\,dt, \qquad \int_0^x h(t)\,dt = \int_0^\infty f(t)\frac{L(xt)}{t}\,dt, \quad (C_1)$$

$$\int_0^x f(t)\,dt = \int_0^\infty g(t)\frac{L(xt)}{t}\,dt = \int_0^\infty h(t)\frac{K(xt)}{t}\,dt, \quad (C_2)$$

† The sufficiency of conditions (B_4), (B_5) was stated, in a slightly different form, by Temple (**8**), whose work appears to have been overlooked by later writers.

it is necessary and sufficient that $K(x)/x$ *and* $L(x)/x$ *should have Mellin transforms* $k(s)/(1-s)$ *and* $l(s)/(1-s)$, *where* $k(s)$ *and* $l(s)$ *are bounded and*

$$k(s)l(1-s) = 1 \quad \textit{for almost all } s. \tag{C$_3$}$$

Further, the transformations given by K *and* L *are bounded,*† *their bounds being respectively* l.u.b. $|k(s)|$ *and* l.u.b. $|l(s)|$. *Finally, if* g_1, g_2 *are the '*K*-transforms' of* f_1, f_2, *then*

$$\int_0^\infty f_1(x)g_2(x)\,dx = \int_0^\infty g_1(x)f_2(x)\,dx \tag{C$_4$}$$

(*with a similar result for* L*-transforms*).

Our object is to exhibit B and C as consequences of some general properties of 'integral transforms', and to bring out the difference in character between them. Our arguments and results could of course be stated in the language of Hilbert-space theory; with one exception (Theorem D, *infra*) we have made no use of that theory, but have preferred to give a more elementary exposition.

3. We shall be dealing with transformations T which assign to every $f \in L^2$ a 'transform' $Tf \in L^2$.

DEFINITION 1. T is called *linear* if, for every pair of real numbers a, b,

$$T(af+bg) = aTf+bTg.$$

T is called *bounded* if there exists a finite constant B such that

$$\int_0^\infty (Tf)^2\,dx \leqslant B^2 \int_0^\infty f^2\,dx$$

for every $f \in L^2$; the least possible value of B is called the *bound* of T.

Any function $\phi(x, y)$ of two variables is called a *kernel*; two kernels ϕ_1, ϕ_2 are regarded as identical if, for each x, $\phi_1(x, y) = \phi_2(x, y)$ for almost all y.

DEFINITION 2. If ϕ is such that to every $f \in L^2$ there corresponds a $g \in L^2$ such that (for all $x \geqslant 0$)

$$\int_0^x g(t)\,dt = \int_0^\infty f(t)\phi(x,t)\,dt, \tag{1}$$

we say that ϕ gives rise to a linear transformation (l.t.) T_ϕ, and write $T_\phi f = g$.

† In the sense of Definition 1, *infra*.

DEFINITION 3. If ϕ has the properties:

(S_1) $\phi(0,t) = 0$ for almost all t;

(S_2) there exists a finite constant B such that, for any k_n, a_n, b_n $(n = 1,\ldots,N;\ 0 \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \ldots \leqslant a_N \leqslant b_N)$, we have

$$\int_0^\infty \Big\{\sum_{n=1}^N k_n\{\phi(b_n,t)-\phi(a_n,t)\}\Big\}^2 dt \leqslant B^2 \sum_{n=1}^N k_n^2(b_n-a_n); \qquad (2)$$

then we call ϕ a *standard kernel*, and the least possible value of B the *modulus* of ϕ.

Observe that, taking $N = 1$, $a_1 = 0$, and $b_1 = x$, (S_1) and (S_2) imply that $\phi(x,t) \in L^2$ for every $x \geqslant 0$.

THEOREM I. *The kernel ϕ gives rise to a bounded l.t. with bound B if and only if ϕ is a standard kernel with modulus B.*

By a theorem due to Riesz (**6**), $\int_0^\infty f(t)\phi(x,t)\,dt$ exists for every $f \in L^2$ if and only if $\phi(x,t) \in L^2$. By another theorem of his, $\int_0^\infty f(t)\phi(x,t)\,dt$ is of the form $\int_0^x g(t)\,dt$ for all $x \geqslant 0$ (with $g \in L^2$) if and only if $\phi(0,t) = 0$ for almost all t and

$$\sum_{n=1}^N \frac{1}{b_n-a_n}\Big(\int_0^\infty f(t)\{\phi(b_n,t)-\phi(a_n,t)\}\,dt\Big)^2 \leqslant \int_0^\infty g^2(t)\,dt, \qquad (3)$$

for every set of non-overlapping intervals (a_n, b_n). It remains to show that the condition

$$\sum_{n=1}^N \frac{1}{b_n-a_n}\Big(\int_0^\infty f(t)\{\phi(b_n,t)-\phi(a_n,t)\}\,dt\Big)^2 \leqslant B^2 \int_0^\infty f^2(t)\,dt \qquad (4)$$

(for all $f \in L^2$) is equivalent to (2) of Definition 3, with the same value of B in each case.

If (4) holds, then we have for every $f \in L^2$

$$\Big(\int_0^\infty f(t)\Big\{\sum_1^N k_n\{\phi(b_n,t)-\phi(a_n,t)\}\Big\}\,dt\Big)^2$$

$$\leqslant \Big[\sum_1^N k_n^2(b_n-a_n)\Big]\Big[\sum_1^N \frac{1}{b_n-a_n}\Big(\int_0^\infty f(t)\{\phi(b_n,t)-\phi(a_n,t)\}\,dt\Big)^2\Big]$$

$$\leqslant B^2 \sum_1^N k_n^2(b_n-a_n).\int_0^\infty f^2(t)\,dt, \quad \text{on using (4).}$$

Then (2) follows by the converse of Schwarz's inequality.

If (2) holds, let c_n $(n = 1,\ldots,N)$ be arbitrary real numbers with $\sum_1^N c_n^2 = 1$. We have then (by the converse of Schwarz's inequality for sums)

$$\sum_1^N \frac{1}{b_n-a_n}\left(\int_0^\infty f(t)\{\phi(b_n,t)-\phi(a_n,t)\}\,dt\right)^2$$

$$= \text{l.u.b.}\left|\sum_1^N c_n \int_0^\infty f(t)\frac{\phi(b_n,t)-\phi(a_n,t)}{(b_n-a_n)^{\frac{1}{2}}}\,dt\right|^2$$

$$\leqslant \int_0^\infty f^2(t)\,dt \cdot \int_0^\infty \left\{\sum_1^N \frac{c_n}{(b_n-a_n)^{\frac{1}{2}}}\{\phi(b_n,t)-\phi(a_n,t)\}\right\}^2 dt$$

$$\leqslant \int_0^\infty f^2(t)\,dt \cdot B^2 \sum_1^N \frac{c_n^2}{b_n-a_n}(b_n-a_n), \quad \text{on using (2)},$$

$$= B^2 \int_0^\infty f^2(t)\,dt, \quad \text{so that (4) holds.}$$

4. Definition 4. Two l.t.'s T, T^* are said to be *adjoint* if, for all $f, g \in L^2$

$$\int_0^\infty Tf.g\,dt = \int_0^\infty f.T^*g\,dt. \tag{5}$$

Definition 5. Two kernels ϕ, ϕ^* are said to be *adjoint* if, for all $x \geqslant 0$, $y \geqslant 0$,

$$\int_0^y \phi(x,t)\,dt = \int_0^x \phi^*(y,t)\,dt. \tag{6}$$

Observe that, with either definition, the adjoint (l.t. or kernel) is unique if it exists.

Theorem II. *If ϕ gives rise to a bounded l.t. T with bound B, then ϕ has an adjoint ϕ^* which gives rise to a bounded l.t. T^* with bound B, adjoint to T.*

Define the simple step-function $s_x(t)$ by

$$s_x(t) = \begin{cases} 1 & (0 \leqslant t \leqslant x), \\ 0 & (t > x). \end{cases} \tag{7}$$

and let $Ts_x(t) = \phi^*(x,t)$. Then $\phi^*(x,t) \in L^2$, and ϕ^* satisfies (6).

Now let $p(t)$ be the step-function which takes values k_n in (a_n, b_n) $(n = 1,..., N)$ and vanishes elsewhere; let $P(t)$ be the similar function defined by K_m, A_m, B_m $(m = 1,..., M)$. It is readily seen that

$$\int_0^\infty P(t)\Big\{\sum_1^N k_n\{\phi^*(b_n, t)-\phi^*(a_n, t)\}\Big\}\,dt$$

$$= \int_0^\infty p(t)\Big\{\sum_1^M K_m\{\phi(B_m, t)-\phi(A_m, t)\}\Big\}\,dt. \quad (8)$$

Hence the square of the left-hand side of (8) does not exceed

$$\int_0^\infty p^2(t)\,dt.\int_0^\infty \Big\{\sum_1^M K_m\{\phi(B_m, t)-\phi(A_m, t)\}\Big\}^2\,dt$$

$$\leqslant \sum_1^N k_n^2(b_n-a_n).B^2\sum_1^M K_m^2(B_m-A_m)$$

$$= B^2\sum_1^N k_n^2(b_n-a_n).\int_0^\infty P^2(t)\,dt,$$

ϕ being a standard kernel with modulus B, by Theorem I. Since $P(t)$ is an arbitrary step-function, it follows from the converse of Schwarz's inequality that ϕ^* satisfies condition (S_2) for a standard kernel with modulus $\leqslant B$, and clearly (S_1) is also satisfied. Therefore, by Theorem I, ϕ^* gives rise to a l.t. T^* with bound $\leqslant B$.

Equation (8) states that condition (5) for the adjointness of T, T^* is satisfied when $f = P$ and $g = p$; since every $f, g \in L^2$ can be arbitrarily closely approximated in mean square by step-functions, and T, T^* are bounded, it follows that (5) must hold for all $f, g \in L^2$, so that T, T^* are adjoint. We have now shown that T, T^* are symmetrically related, and hence their bounds must be equal (since we proved above that bound $T^* \leqslant$ bound T).

THEOREM III. *If T is a bounded l.t. it has an adjoint T^*, and T, T^* are given by adjoint kernels.*†

We define $s_x(t)$, $\phi^*(x, t)$, and $p(t)$ as in the proof of Theorem II. It follows that

$$Tp(t) = \sum_1^N k_n\{\phi^*(b_n, t)-\phi^*(a_n, t)\},$$

† The first part of this theorem is well known in Hilbert-space theory; the second part was proved by Bochner (1).

so that

$$\int_0^\infty \left\{\sum_1^N k_n\{\phi^*(b_n,t)-\phi^*(a_n,t)\}\right\}^2 dt = \int_0^\infty \{Tp(t)\}^2\,dt$$

$$\leqslant B^2 \int_0^\infty p^2(t)\,dt = B^2 \sum_1^N k_n^2(b_n-a_n),$$

where B is the bound of T. Since $\phi^*(0,t)=0$, ϕ^* is a standard kernel with modulus $\leqslant B$ and (by Theorem I) gives rise to a l.t. T' with bound $\leqslant B$. By Theorem II, ϕ^* has an adjoint ϕ giving rise to a l.t. T'' (adjoint to T') with bound $\leqslant B$. But, as in the proof of Theorem II, $\phi^*(x,t) = T''s_x(t)$; hence T, T'' coincide for all step-functions, and, since they are both bounded, they must be identical. The theorem follows on choosing $T^* = T'$.

5. We state now the following result from Hilbert-space theory.†

THEOREM D. *If a l.t. T has an adjoint T^*, then both T and T^* are bounded.*

(It should be emphasized here that both Tf and T^*f must be defined and belong to L^2 for *all* $f \in L^2$.)

This result will be required later in proving Theorem E; for the present, we merely note the following consequence.

THEOREM IV. *If ϕ has an adjoint ϕ^*, and both ϕ and ϕ^* give rise to l.t.'s, then both are standard kernels (and the l.t.'s are bounded).*

6. DEFINITION 6. Given two l.t.'s T_1, T_2 we say that T_2 is a *left inverse* of T_1 if, for every $f \in L^2$, $T_2(T_1 f) = f$. This clearly implies that T_1 is $(1,1)$, and it is easily seen that T_2 is also a right inverse of T_1 if and only if T_2 is $(1,1)$. If this is the case, we say that T_1, T_2 are *inverse*, and their bounds B_1, B_2 then satisfy $B_1 B_2 \geqslant 1$.

DEFINITION 7. Two kernels ϕ, ψ are called *biorthogonal* if, for all $x \geqslant 0$, $y \geqslant 0$,

$$\int_0^\infty \phi(x,t)\psi(y,t)\,dt = \min(x,y). \tag{9}$$

Observe that, if ϕ, ψ are standard kernels, this definition is consistent with $\phi(0,t) = \psi(0,t) = 0$ for almost all t.

THEOREM V. *Let ϕ, ψ be standard kernels, giving rise to l.t.'s T_ϕ, T_ψ. Then T_ψ is a left inverse of T_ϕ if and only if ϕ^* and ψ are biorthogonal.*

† Stone (7), Theorem 2.26.

COROLLARY. *Under these circumstances T_{ϕ^*} is a left inverse of T_{ψ^*} and T_ϕ, T_{ψ^*} are $(1,1)$. T_ϕ and T_ψ (or T_{ϕ^*} and T_{ψ^*}) are now inverse if and only if one of T_ψ, T_{ϕ^*} is $(1,1)$ (in which case the other is also), and then ϕ and ψ^* are biorthogonal.*

We note that $T_\phi s_x(t) = \phi^*(x,t)$ and hence $T_\psi \phi^*(x,t) = s_x(t)$,

i.e. $$\int_0^\infty \phi^*(x,t)\psi(y,t)\,dt = \int_0^y s_x(t)\,dt = \min(x,y).$$

This proves the necessity; to prove the sufficiency, note again that $T_\phi s_x(t) = \phi^*(x,t)$ and that now $T_\psi \phi^*(x,t) = s_x(t)$ because ϕ^*, ψ are biorthogonal. Hence $T_\psi(T_\phi f) = f$ if f is a step-function; since T_ϕ, T_ψ are bounded, this implies that $T_\psi(T_\phi f) = f$ for all $f \in L^2$.

The corollary follows at once from the remarks that were made after Definition 6.

7. We are now in a position to discuss Theorems B and C (stated in the Introduction).

Theorem B is a straightforward deduction from our results. (B_1) and (B_2) state that T_ϕ and T_ψ are inverse l.t.'s, and (B_3), applied to f, g, h, and F (with $T_\phi F = G$, $T_\psi F = H$), yields the more general relation

$$\int_0^\infty fF\,dt = \int_0^\infty gG\,dt = \int_0^\infty hH\,dt.$$

This is readily seen to be equivalent to the adjointness of T_ϕ, T_ψ or (by Theorem III) to the adjointness of ϕ, ψ, which is (B_4). Moreover, by Theorem V, T_ϕ and T_ψ are inverse if and only if ϕ^*, ψ and ψ^*, ϕ are biorthogonal, which is (B_5) if (B_4) is taken into account. Observing finally that (B_4) implies that ϕ, ψ are standard kernels (with modulus $\leqslant 1$), Theorem B follows at once. (Note that since the l.t.'s are given as bounded we have not had to use Theorem D.)

In Hilbert-space terminology, Theorem B deals with an inverse pair of unitary† transformations. On the other hand, the following theorem (which includes Theorem C) deals with an inverse pair of self-adjoint transformations.

† Since we deal only with real functions and transformations, we may also call the transformations orthogonal.

THEOREM E. *Suppose that to every $f \in L^2$ there correspond $g, h \in L^2$ such that (for all $x \geqslant 0$)*

$$\int_0^x g(t)\,dt = \int_0^\infty f(t)\phi(x,t)\,dt, \qquad \int_0^x h(t)\,dt = \int_0^\infty f(t)\psi(x,t)\,dt, \tag{E_1}$$

$$\int_0^x f(t)\,dt = \int_0^\infty g(t)\psi(x,t)\,dt = \int_0^\infty h(t)\phi(x,t)\,dt. \tag{E_2}$$

Let $G, H \in L^2$ correspond in the same way to $F \in L^2$, and suppose that we must then have

$$\int_0^\infty gF\,dt = \int_0^\infty fG\,dt, \qquad \int_0^\infty hF\,dt = \int_0^\infty fH\,dt. \tag{E_3}$$

For all this, it is necessary and sufficient that ϕ, ψ be standard kernels, and that (for all $x \geqslant 0$, $y \geqslant 0$)

$$\int_0^\infty \phi(x,t)\psi(y,t)\,dt = \min(x,y), \tag{E_4}$$

$$\int_0^y \phi(x,t)\,dt = \int_0^x \phi(y,t)\,dt, \qquad \int_0^y \psi(x,t)\,dt = \int_0^x \psi(y,t)\,dt. \tag{E_5}$$

The theorem remains true if (E_3) *is deleted and* (E_5) *is made part of the hypotheses. Finally, the moduli B_1, B_2 of ϕ, ψ satisfy $B_1 B_2 \geqslant 1$.*

Conditions (E_1), (E_2) state that T_ϕ, T_ψ are inverse l.t.'s; (E_3) then states that T_ϕ, T_ψ are self-adjoint, i.e. that $T_\phi^* = T_\phi$, $T_\psi^* = T_\psi$.

To prove necessity, observe that the above conditions imply that T_ϕ, T_ψ are bounded (by Theorem D); hence that ϕ, ψ are standard kernels (by Theorem I). Theorem III now shows that $\phi = \phi^*$, $\psi = \psi^*$, which is (E_5), and then Theorem V shows that (E_4) is satisfied. If (E_5) is given instead of (E_3), we appeal to Theorem IV.

The sufficiency proof is straightforward, and does not require any appeal to Theorem D. (Incidentally, the use of Theorem D may be avoided if we are given that T_ϕ, T_ψ are bounded.)

Now take $\phi(x,t) = K(xt)/t$ and $\psi(x,t) = L(xt)/t$. If we define (as we may) $K(0) = L(0) = 0$, (E_5) is automatically satisfied, and (E_3) becomes a separate deduction. To deduce Theorem C, we need only observe that (C_3) and (E_4) are equivalent (by Parseval's formula for Mellin transforms) and it then remains to prove the following

LEMMA. *$K(xt)/t$ is a standard kernel if and only if $k(s)$ is bounded, and then the modulus of the kernel is equal to* l.u.b. $|k(s)|$.

The Mellin transform of

$$\sum_1^N k_n \frac{K(b_n t)-K(a_n t)}{t}$$

is

$$\sum_1^N k_n(b_n^{-s}-a_n^{-s})\frac{k(s)}{1-s} = k(s)\overline{\mathfrak{F}_N(s)},$$

where $\mathfrak{F}_N(s)$ is the Mellin transform of the step-function $f_N(t)$ which takes values k_n in (a_n, b_n) and vanishes elsewhere. By Parseval's formula

$$\int_0^\infty \left\{\sum_1^N k_n \frac{K(b_n t)-K(a_n t)}{t}\right\}^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |k(s)|^2 |\mathfrak{F}_N(s)|^2 \, d\tau,$$

and

$$\sum_1^N k_n^2(b_n-a_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathfrak{F}_N(s)|^2 \, d\tau.$$

Hence $K(xt)/t$ is a standard kernel (with modulus $\leqslant B$) if and only if

$$\int_{-\infty}^{\infty} |k(s)|^2 |\mathfrak{F}_N(s)|^2 \, d\tau \leqslant B^2 \int_{-\infty}^{\infty} |\mathfrak{F}_N(s)|^2 \, d\tau \qquad (10)$$

for all $\mathfrak{F}_N(s)$. Since the $f_N(t)$ are dense in L^2, the $\mathfrak{F}_N(s)$ are dense in $\mathfrak{L}^2$ (again by Parseval's formula), so that (10) will hold if and only if l.u.b. $|k(s)| \leqslant B$. The argument clearly proves that B must have the same value in both cases.

We see then that what Kober calls a 'Plancherel–Watson transformation' is a special case of the self-adjoint transformation of Theorem E, whilst the Bochner–Kaczmarz theorem (Theorem B) deals with orthogonal transformations. A transformation can belong to both classes only if it is involutory, i.e. if $T^2 f = f$, and this is precisely the case dealt with by Theorem A (for kernels of the special form $K(xt)/t$). But here Theorem E allows us to drop the orthogonality condition $TT^* = I$ (the identity transformation), since this is implied by $T^* = T$ and $T^2 = I$.

By putting $\phi = \psi$ in Theorem E, one obtains the general case of an involutory l.t., which stands in the same relation to Theorem E as Theorem A does to Theorem C. The condition that ϕ be a standard kernel can now be omitted since it is implied by the orthogonality of T_ϕ. The precise formulation of the theorem is left

to the reader. If we call it Theorem F, and observe that it can easily be deduced from B, we see that the implications shown in the following diagram hold.

$$\begin{array}{ccc} & & \mathrm{E} \to \mathrm{C} \\ & & \downarrow \quad \downarrow \\ \mathrm{B} & \to & \mathrm{F} \quad \mathrm{A} \end{array}$$

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A CONDITION UNDER WHICH AN IRREDUCIBLE IDEAL IS PRIMARY

By L. FUCHS (*Budapest*)

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LET there be given a general abstract commutative ring $\mathfrak{R}$. It is a well-known fact that, if in $\mathfrak{R}$ the ascending-chain condition holds, then every irreducible ideal in $\mathfrak{R}$ is at the same time primary.* This fact is of basic importance in the theory of ideals, since it admits the proof of the existence of a primary representation for every ideal of $\mathfrak{R}$. Certain questions have been treated which concern the connexion between irreducible and primary ideals;† it seems, however, that one interesting and rather important question has been left unanswered, namely, the question of determining a necessary and sufficient condition under which an irreducible ideal of $\mathfrak{R}$ is primary. The purpose of the present note is to determine such a condition.

The usual proof that every irreducible ideal is primary‡ shows that not only the ascending-chain condition implies this fact, but so does also the weaker

QUOTIENT-CHAIN CONDITION.§ *Every quotient-chain of ideals,* $\mathfrak{a} \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots$, *where* $\mathfrak{a}_i$ *is of the form* $\mathfrak{a}_{i-1}:\mathfrak{b}_i$ *and is a proper over-ideal of* $\mathfrak{a}_{i-1}$, *breaks off after a finite number of terms.*

Our stated problem will be solved by a weaker form of the quotient-chain condition. In what follows a chain $\mathfrak{a} \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots$ is said to be a *principal quotient-chain* if $\mathfrak{a}_1 = \mathfrak{a}:(b)$ and in general‖

$$\mathfrak{a}_k = \mathfrak{a}_{k-1}:(b) = \mathfrak{a}:(b^k)$$

for every k. I shall say that *for the ideal* $\mathfrak{a}$ *the principal quotient-chain condition holds,* if every principal quotient-chain with the first term $\mathfrak{a}$ terminates. I now formulate the main theorem.

THEOREM 1. *The necessary and sufficient condition that an irreducible ideal* $\mathfrak{q}$ *be primary is that for* $\mathfrak{q}$ *the principal quotient-chain condition holds.*

I begin by proving the *necessity* of the stated condition. Given an

* See e.g. Noether (**1**), 39, Krull (**2**), 9, Waerden (**1**), 31.

† See Gröbner (**1**). ‡ See in particular the proof of Waerden (**1**), 31.

§ This condition is due to W. Krull; see his paper (**1**). The sign $\subseteq$ means inclusion, while $\subset$ means proper inclusion.

‖ For the rules on forming ideal quotients see e.g. Waerden (**1**), 25.

element b, there are two cases to distinguish. It may happen that b is contained in the prime radical* $\mathfrak{g}$ of the primary ideal $\mathfrak{q}$ and then has a power b^n contained in $\mathfrak{q}$. Therefore†

$$\mathfrak{q}:(b^n) = \mathfrak{q}:(b^{n+1}) = \ldots = \mathfrak{v}.$$

Or again it may happen that b is not in $\mathfrak{g}$ and is consequently *prime to*‡ $\mathfrak{q}$: that is to say,

$$\mathfrak{q} = \mathfrak{q}:(b) = \mathfrak{q}:(b^2) = \ldots.$$

Both cases prove that for $\mathfrak{q}$ the principal quotient-chain condition holds.

For the proof of the *sufficiency* of the indicated condition we need a simple lemma on irreducible ideals.

LEMMA. *Any two proper over-ideals of an irreducible ideal* $\mathfrak{q}$ *contain a common element* c *which is not contained in* $\mathfrak{q}$ *itself.*

Indeed, in the contrary case $\mathfrak{q}$ would be reducible.

Resuming the proof of the theorem, we take an element a not contained in the irreducible ideal $\mathfrak{q}$ and an element b no power of which is contained in $\mathfrak{q}$. In order to establish that $\mathfrak{q}$ is primary, I shall show that ab cannot be contained in $\mathfrak{q}$.

Since the principal quotient-chain condition is now assumed to hold for $\mathfrak{q}$, the chain of the ideal quotients $\mathfrak{q} \subset \mathfrak{q}:(b) \subset \mathfrak{q}:(b^2) \subset \ldots$ must terminate, so that for a certain n we have $\mathfrak{q}:(b^n) = \mathfrak{q}:(b^{n+1})$. The ideals $\mathfrak{q}+(a)$ and $\mathfrak{q}+\mathfrak{v}.b^n$ are proper over-ideals of $\mathfrak{q}$, and therefore, by the lemma, both must contain a common element c which does not belong to $\mathfrak{q}$;

$$c = q_1+ra+ma = q_2+sb^n$$

with q_1, q_2 in $\mathfrak{q}$, r, s in $\mathfrak{R}$, and m a rational integer. If now the element ab were contained in $\mathfrak{q}$, we should get

$$bc = q_1b+rab+mab = q_2b+sb^{n+1} \subset \mathfrak{q}$$

and hence $sb^{n+1} \subset \mathfrak{q}$. This shows that $s \subset \mathfrak{q}:(b^{n+1})$, and hence, by the choice of n, also $s \subset \mathfrak{q}:(b^n)$. Nevertheless $sb^n \subset \mathfrak{q}$ contradicts the fact that $c = q_2+sb^n$ is not contained in $\mathfrak{q}$, establishing that $\mathfrak{q}$ is indeed primary.§

* The radical consists of the elements which have a power contained in $\mathfrak{q}$.

† $\mathfrak{v}$ denotes the unit ideal of $\mathfrak{R}$. ‡ Waerden (**1**), 29.

§ In spite of the great similarity between the proof given here and van der Waerden's proof there is, however, an essential difference which lies in the fact that we need the principal quotient-chain condition for *irreducible* ideals, while van der Waerden makes use of the same condition (not expressly stated, of course) for *non-primary* ideals.

As a corollary to Theorem 1 we get

THEOREM 2. *Every irreducible ideal of $\mathfrak{R}$ is primary if and only if for every irreducible ideal of $\mathfrak{R}$ the principal quotient-chain condition holds.*

One simple remark on the quotient-chain condition may finally be inserted here. That the ascending-chain condition implies the quotient-chain condition is seen immediately. It is, however, not so evident that the same is true for the descending-chain condition.* In order to prove this assertion, let us consider the descending chain

$$\mathfrak{b}_1+\mathfrak{a} \supseteq \mathfrak{b}_1\mathfrak{b}_2+\mathfrak{a} \supseteq \mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3+\mathfrak{a} \supseteq \dots,$$

where all members are over-ideals of $\mathfrak{a}$, and which contains by hypothesis only a finite number of different terms. In the quotient chain $\mathfrak{a} \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$, where

$$\mathfrak{a}_i = \mathfrak{a}_{i-1}:\mathfrak{b}_i = (\mathfrak{a}_{i-2}:\mathfrak{b}_{i-1}):\mathfrak{b}_i = \mathfrak{a}_{i-2}:(\mathfrak{b}_{i-1}\mathfrak{b}_i) = \mathfrak{a}:(\mathfrak{b}_1\dots\mathfrak{b}_i)$$
$$= \mathfrak{a}:(\mathfrak{b}_1\dots\mathfrak{b}_i+\mathfrak{a}),$$

from a certain point on all ideals are therefore in fact equal.

* For the descending-chain condition see Noether (**2**). By descending-chain condition is meant here 'der abgeschwächte Vielfachenkettensatz'.

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SUMMATIONS OVER LATTICE POINTS IN κ-SPACE

By S. BOCHNER and K. CHANDRASEKHARAN (*Princeton*)

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1. Introduction

In the analytic theory of numbers it is an outstanding problem to determine the precise order of the arithmetical function $P(x)$, which is defined as follows. If

$$r(n) = \sum_{n_1^2+n_2^2=n} 1$$

for integral values of n_1 and n_2 and

$$R(x) = \sum_{n \leqslant x} r(n),$$

then

$$R(x) = \pi x + P(x).$$

Hardy's conjecture [5] that $P(x) = O(x^{\frac{1}{4}+\epsilon})$ for every positive number ϵ is obviously based on a consideration of the behaviour of the expansion of $R^\alpha(x)$ in a series of Bessel functions, where

$$R^\alpha(x) = \sum_{n \leqslant x} (x-n)^\alpha r(n) \quad (\alpha > 0).$$

If x is positive and non-integral and $\alpha \geqslant 0$, then Hardy proves [5]

$$R^\alpha(x) = \frac{\pi x^{1+\alpha}}{1+\alpha} - x^\alpha - \pi^{-\alpha}\Gamma(1+\alpha)x^{\frac{1}{2}+\frac{1}{2}\alpha} \sum \frac{r(n)J_{1+\alpha}\{2\pi\sqrt{(nx)}\}}{n^{\frac{1}{2}+\frac{1}{2}\alpha}}. \tag{1.1}$$

If $\alpha = 0$, then the definition of $R(x)$ must be given the usual modification when x is an integer, namely, the last term $r(x)$ in the sum $\sum_{n \leqslant x} r(n)$ is to be replaced by $\frac{1}{2}r(x)$. The identity (1.1) has been the starting-point for similar results in higher dimensions and more general domains than the circle $x^2+y^2 = n$. Wilton, for instance, has considered summations over lattice points, with or without weights, in κ-dimensional ellipsoids [**11**, **12**]. Instead of convergent expansions like (1.1) one expects in such cases summable series; there will also arise a distinction in behaviour between the two cases $\alpha \geqslant 0$ and $\alpha < 0$. Wilton's results, for the circle, have been considerably sharpened by Dixon and Ferrar [**4**], who have given a rather powerful method based on Landau's summation-formula involving $r(n)$ [**8**, 274, Satz 559] together with some estimates for integrals of Bessel functions and a theorem of Ananda-Rau on Dirichlet's series [**1**, Th. 7].

(Their results settle a point raised much later by Hardy perhaps by oversight.)†

In reviewing all these summation-formulae, we have been led to the conclusion that they arise in a simple and natural way from a consideration of the spherical summation of multiple Fourier series. We accordingly start with a function $f(x_1,\ldots,x_\kappa)$ defined over the whole of κ-space, belonging to the Lebesgue class L, and vanishing outside a suitable domain, for example, a hyper-sphere, and then construct, in an obvious manner, a periodic function of class L (or an almost periodic function of Stepanoff class, as the case may be) whose Fourier series at the origin is exactly the series occurring in lattice-point problems. Our results include those previously known; in addition, we obtain, what the earlier papers did not include, summability-theorems for series of the form

$$\sum \frac{r_\kappa(n,h)J_{\alpha+\frac{1}{2}\kappa}\{2\pi\sqrt{(nx)}\}}{n^\gamma}$$

with γ not necessarily equal to $\frac{1}{2}\alpha+\frac{1}{4}\kappa$, where

$$r_\kappa(n,h)=\sum_{n_1^2+\ldots+n_\kappa^2=n} e^{2\pi i(n_1h_1+\ldots+n_\kappa h_\kappa)},$$

so that $r_\kappa(n,0)=r_\kappa(n)$ if $h_1=h_2=\ldots=h_\kappa=0$, and $r(n)\equiv r_2(n)$.

More generally, we replace $r_\kappa(n,h)$ by $r_\kappa^P(\lambda_n,h)$ where

$$P=P(n,n)=\sum_{j,l=1}^{\kappa} a_{jl}n_jn_l$$

is a positive definite symmetric form, and

$$r_\kappa^P(n,h)=\sum_{P(n,n)=\lambda_n} e^{2\pi i(n_1h_1+\ldots+n_\kappa h_\kappa)},$$

where λ_n is the nth number occurring in the sequence of values of $P(n,n)$ when they are arranged in increasing order of magnitude.

The application of our results to the evaluation of the average order of the arithmetical function $r_\kappa(n,h)$ will be considered in another paper.

The inter-relations between Poisson's summation-formula and summations over lattice-points have been clearly recognized by Landau [9] and Müntz [10] and explicitly brought out by Bochner [3]; other pertinent references are mentioned in Wilton's papers [11, 12]. In Wilton's paper [11] the summation is not *multiple* but

† See (6) 82, line 12.

repeated, and, as was emphatically pointed out by Hardy in a footnote [**11**, 228], the difference between the two is very considerable. In Wilton [**12**] a result resembling ours does occur [Th. D′, 181] but our conclusions are much more comprehensive, including as they do most of the conclusions of Dixon and Ferrar as well [**4**].

2. Preliminary lemmas

LEMMA 1. *If $J_\mu(x)$ denotes the Bessel function of the first kind and of order μ, and $V_\mu(x) = J_\mu(x)/x^\mu$, then*

$$V_\nu(xR) = \frac{2^{\mu+1-\nu}}{\Gamma(\nu-\mu)x^{2\nu}} \int_0^x (x^2-y^2)^{\nu-\mu-1} y^{2\mu+1} V_\mu(yR)\, dy$$

for $\nu > \mu > -1$ [**13**, 372].

LEMMA 2. *For a fixed $\rho > -1$,*

$$z^{\frac{1}{2}} J_\rho(z) \sim \cos z \sum_{n=0}^{m-1} A_n z^{-n} + \sin z \sum_{n=0}^{m-1} B_n z^{-n} + O(z^{-m}),$$

where A_n, B_n are suitable constants. Hence

$$z^{-\frac{1}{2}} e^{iz} = P_0(z^{-1}) J_\rho(z) + P_1(z^{-1}) J_{\rho+\epsilon}(z) + O(z^{-m-\frac{1}{2}})$$

for a fixed ϵ $(0 < \epsilon < 1)$ and large z, where

$$P_0(z^{-1}) = \sum_{r=0}^{m-1} \alpha_r z^{-r}, \qquad P_1(z^{-1}) = \sum_{r=0}^{m-1} \beta_r z^{-r},$$

and α_r, β_r are suitable constants.

The lemma is obvious from the asymptotic expansion of J_ρ [**13**, 199].

LEMMA 3. *If $\sum a_n$ is summable by Riesz's means of type n and order r—briefly, summable $(R; n, r)$—and $a_n = O(n^\alpha)$ and $0 \leqslant l < r$, then $\sum a_n/n^\sigma$ is summable $(R; n, l)$ for*

$$\sigma > \frac{(\alpha+1)(r-l)}{r+1}.$$

This is a special case of a theorem of Ananda Rau [**1**, Th. 7].

LEMMA 4. *If $\sigma_\lambda(\kappa)$ denotes the abscissa of summability $(R; \lambda, k)$ of the Dirichlet series*

$$\sum a_n e^{-\lambda_n s},$$

then $\sigma_\lambda(\kappa)$ is a convex function of k.

This is a theorem of M. Riesz and H. Bohr [**7**, 17].

LEMMA 5. *If $f(x) \equiv f(x_1,\dots,x_\kappa)$ is a periodic function of class L, or more generally, an almost periodic function of Stepanoff class (in which case the $\{n_\kappa\}$ need not be integers) and*

$$f(x) \sim \sum_n a(n) e^{i\Lambda(n,x)},$$

where $\Lambda(n,x)$ denotes $n_1x_1+\dots+n_\kappa x_\kappa$, and $a(n) = a(n_1,\dots,n_\kappa)$ is the Fourier coefficient, and $D^q(n_1,\dots,n_\kappa)$ is, for any non-negative integer q, a homogeneous polynomial of total degree q in $n_1,\dots,n_\kappa$, then

(i) *the operator* $$D_x^q = D^q\left(\frac{\partial}{i\partial x_1},\dots,\frac{\partial}{i\partial x_\kappa}\right)$$

applies to the almost periodic function

$$S_R^\delta(x) = \sum_{|n|\leqslant R}\left(1-\frac{|n|^2}{R^2}\right)^\delta a(n)e^{i\Lambda(n,x)}$$

and the resulting function is almost periodic;

(ii) $$D_x^q S_R^\delta(x) = \sum_{|n|\leqslant R}\left(1-\frac{|n|^2}{R^2}\right)^\delta a(n) D_n^q e^{i\Lambda(n,x)};$$

(iii) *for every point x at which $f(x)$ is differentiable q times, we have*

$$\lim_{R\to\infty}[D_x^q S_R^\delta(x) - D_x^q f(x)] = 0$$

for $\delta > \frac{1}{2}(\kappa-1)+q$.

This is a theorem of Bochner [**2**, Th. 1], stated here with a slight modification.

LEMMA 6. *Let $\beta > -1$ and*

$$g(x_1,\dots,x_\kappa) = \begin{cases} \left[\xi^2 - \left(\sum_{j=1}^\kappa x_j^2\right)\right]^\beta & \text{if } \sum x_j^2 < \xi^2, \\ 0 & \text{if } \sum x_j^2 \geqslant \xi^2. \end{cases}$$

Let $$f(x_1,\dots,x_\kappa) = \sum_{p_\kappa=-\infty}^{\infty}\cdots\sum g(x_1+p_1,\dots,x_\kappa+p_\kappa)$$

where $\{p_\kappa\}$ are integers. Let

$$a_{n_1,\dots,n_\kappa} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\cdots\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_1,\dots,x_\kappa)e^{-2\pi i\sum n_j x_j}\,dx_1\dots dx_\kappa.$$

Then

$$a_{n_1,\dots,n_\kappa} = \frac{\xi^{\beta+\frac{1}{2}\kappa}\Gamma(\beta+1)\pi^{-\beta}J_{\beta+\frac{1}{2}\kappa}\{2\pi\xi\sqrt{(n_1^2+\dots+n_\kappa^2)}\}}{(n_1^2+\dots+n_\kappa^2)^{\frac{1}{4}\beta+\frac{1}{4}\kappa}},$$

when $n_1,\dots,n_\kappa$ are not all zero, and

$$a_{0,\dots,0} = \frac{\xi^{2\beta+\kappa}\Gamma(\beta+1)\pi^{\frac{1}{2}\kappa}}{\Gamma(\beta+1+\frac{1}{2}\kappa)}.$$

Proof. It is easily seen that

$$a_{n_1,\dots,n_\kappa} = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} g(x_1,\dots,x_\kappa)e^{-2\pi i\sum n_j x_j}\,dx_1\dots dx_\kappa.$$

Since g vanishes outside $\sum x_j^2 < \xi^2$, we have

$$\begin{aligned} a_{n_1,\dots,n_\kappa} &= \int_{\sum x_j^2<\xi^2}\dots\int (\xi^2-\textstyle\sum x_j^2)^\beta e^{-2\pi i\sum n_j x_j}\,dx_1\dots dx_\kappa \\ &= (2\pi)^{\frac{1}{2}\kappa}\int_0^\xi (\xi^2-t^2)^\beta t^{\kappa-1}V_{\frac{1}{2}\kappa-1}\{2\pi t\sqrt{(n_1^2+\dots+n_\kappa^2)}\}\,dt \\ &= (2\pi)^{\frac{1}{2}\kappa}V_{\beta+\frac{1}{2}\kappa}\{2\pi\xi\sqrt{(n_1^2+\dots+n_\kappa^2)}\}\xi^{2\beta+1}\Gamma(\beta+1)2^\beta, \end{aligned}$$

by Lemma 1. On the other hand,

$$a_{0,\dots,0} = \frac{(2\pi)^{\frac{1}{2}\kappa}}{\Gamma(\frac{1}{2}\kappa)}\int_0^\xi (\xi^2-t^2)^\beta t^{\kappa-1}\,dt = \frac{\xi^{2\beta+\kappa}\Gamma(\beta+1)\Gamma(\frac{1}{2}\kappa)2\pi^{\frac{1}{2}\kappa}}{2\Gamma(\beta+1+\frac{1}{2}\kappa)\Gamma(\frac{1}{2}\kappa)}$$

which proves the lemma.

3. Summability theorems

Consider the function $F(x_1,\dots,x_\kappa)$

and a linear transformation T defined by

$$T:\quad y_\lambda = \sum_{p=1}^{\kappa} a_{\lambda p}x_p \quad (\lambda = 1,\dots,\kappa)$$

with positive determinant

$$D = |a_{\lambda p}| > 0.$$

T transforms F into G, say; then we have

$$T[F(x_1,\dots,x_\kappa)] = G(y_1,\dots,y_\kappa).$$

Consider the translated function $G(y_1+l_1,\dots,y_\kappa+l_\kappa)$, where $\{l_\kappa\}$ are integers. Let

$$f(y_1,\dots,y_\kappa) = \sum_{l_j=-\infty}^{\infty}\dots\sum G(y_1+l_1,\dots,y_\kappa+l_\kappa)$$

and

$$\phi(n_1,\dots,n_\kappa) = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} G(y_1,\dots,y_\kappa)e^{-2\pi i\sum n_p y_p}\,dy_1\dots dy_\kappa.$$

Then

$$\phi(n_1,...,n_\kappa) = D\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} F(x_1,...,x_\kappa)e^{-2\pi i\sum_p x_p\left(\sum_j a_{jp}n_j\right)}\,dx_1...dx_\kappa$$

$$= D\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} F(x_1,...,x_\kappa)e^{-2\pi i\sum x_p N_p}\,dx_1...dx_\kappa,$$

where $$N_p = T^*(n_p) = \sum_{j=1}^{\kappa} a_{jp}n_j.$$

Now let us choose

$$F(x_1,...,x_\kappa) = \begin{cases}[\xi^2-(x_1^2+...+x_\kappa^2)]^\beta & \text{if } \sum x_\kappa^2 < \xi^2,\\ 0 & \text{if } \sum x_j^2 \geqslant \xi^2,\end{cases}$$

where $\beta > -1$. Then, when $n_1,...,n_\kappa$ are not all zero, we have

$$\phi(n_1,...,n_\kappa) = D\int_{\sum x_j^2<\xi^2}...\int (\xi^2-\sum x_j^2)^\beta e^{-2\pi i\sum x_p N_p}\,dx_1...dx_\kappa$$

$$= \frac{D\xi^{2\beta+\kappa}\Gamma(\beta+1)\pi^{-\beta}J_{\beta+\frac{1}{2}\kappa}\{2\pi\xi\sqrt{(N_1^2+...+N_\kappa^2)}\}}{(N_1^2+...+N_\kappa^2)^{\frac{1}{2}\beta+\frac{1}{4}\kappa}} \qquad (3.1)$$

and

$$\phi(0,...,0) = \frac{D\xi^{2\beta+\kappa}\Gamma(\beta+1)\pi^{\frac{1}{2}\kappa}}{\Gamma(\beta+1+\frac{1}{2}\kappa)}$$

as in Lemma 6.

Since $f(y_1,...,y_\kappa)$ is periodic in each variable with period 1, and since it is integrable L (on account of the choice of F), it has a Fourier series:

$$f(y_1,...,y_\kappa) \sim \sum_{(n_\kappa)}...\sum C_{n_1...n_\kappa}e^{2\pi i(n_1y_1+...+n_\kappa y_\kappa)}, \qquad (3.2)$$

and, as in Lemma 5, we easily see that

$$C_{n_1,...,n_\kappa} = \phi(n_1,...,n_\kappa). \qquad (3.3)$$

Hence

$$f(y_1,...,y_\kappa) \sim A+B\sum_{(n_\kappa)}...\sum{}' \frac{J_{\beta+\frac{1}{2}\kappa}\{2\pi\xi\sqrt{(N_1^2+...+N_\kappa^2)}\}e^{2\pi i\sum n_j y_j}}{(N_1^2+...+N_\kappa^2)^{\frac{1}{2}\beta+\frac{1}{4}\kappa}}, \qquad (3.4)$$

where $A = \phi(0,...,0)$ and $B = D\xi^{2\beta+\kappa}\Gamma(\beta+1)\pi^{-\beta}$ and $\sum...\sum'$ stands for summation with the omission of the point $n = (0,...,0)$.

Let $$P(x,x) = \sum_{l,j=1}^{\kappa} b_{jl}x_j x_l \qquad (3.5)$$

stand for a *given* positive definite symmetric form. We may write P as

$$P(x,x) = \sum_{j=1}^{\kappa}\Big(\sum_{l=1}^{\kappa} a_{jl}x_l\Big)^2 \qquad (3.6)$$

and define T by the numbers a_{jl} as given in (3.6). Now form $P(n,n)$ for all possible integers $(n_1,\dots,n_\kappa)$; let the set of values of $P(n,n)$ be denoted by

$$\lambda_0 < \lambda_1 < \dots < \lambda_n < \dots .$$

We may then rewrite (3.4) as follows:

$$f(y_1,\dots,y_\kappa) \sim A+B\sum \dots \sum' \frac{J_{\beta+\frac{1}{2}\kappa}(2\pi\xi\sqrt{\lambda_n})e^{2\pi i\sum n_j y_j}}{\lambda_n^{\frac{1}{2}\beta+\frac{1}{4}\kappa}} \tag{3.7}$$

$$\text{or} \quad f(Tx_1,\dots,Tx_\kappa) \sim A+B\sum \dots \sum' \frac{J_{\beta+\frac{1}{2}\kappa}(2\pi\xi\sqrt{\lambda_n})e^{2\pi i\sum N_i x_j}}{\lambda_n^{\frac{1}{2}\beta+\frac{1}{4}\kappa}}, \tag{3.8}$$

where N_j is as defined above.

If we sum spherically, we may rewrite the right side of (3.8) as

$$A+B\sum_{m=1}^{\infty} \frac{r_\kappa^P(\lambda_m,h)J_{\beta+\frac{1}{2}\kappa}(2\pi\xi\sqrt{\lambda_m})}{\lambda_m^{\frac{1}{2}\beta+\frac{1}{4}\kappa}}, \tag{3.9}$$

where

$$r_\kappa^P(\lambda_n,x) = \sum_{P(n,n)=\lambda_n} e^{2\pi i(n_1x_1+\dots+n_\kappa x_\kappa)}. \tag{3.10}$$

The left side of (3.8), on the other hand, is equal to

$$\sum_{Q(h+l,h+l)<\xi^2} [\xi^2-Q(h+l,h+l)]^\beta, \tag{3.11}$$

where

$$Q(x,x) = \sum_{r,s=1}^{\kappa} b_{rs}x_r x_s$$

and (b_{rs}) is the inverse of $(a_{\lambda p})$.

We now apply the summability-theorem of Lemma 5, with $q=0$, to the Fourier series in (3.8) taken in conjunction with (3.9) and (3.11), and obtain the following

THEOREM I.

$$\sum_{Q(h+l,h+l)<\xi^2} [\xi^2-Q(h+l,h+l)]^\beta = \lim_{R\to\infty}\left[A+B\sum_{m=1}^{\infty}\left(1-\frac{\lambda_m}{R^2}\right)^{\frac{1}{2}(\kappa-1)+\epsilon} \frac{r_\kappa^P(\lambda_m,h)J_{\beta+\frac{1}{2}\kappa}(2\pi\xi\sqrt{\lambda_m})}{\lambda_m^{\frac{1}{2}\beta+\frac{1}{4}\kappa}}\right] \tag{3.12}$$

for every $\epsilon>0$ *and* $\beta>-1$.

Let us consider now the differential operator [cf. (3.6)]:

$$P_x = \sum_{j=1}^{\kappa}\left(\sum_{l=1}^{\kappa} a_{lj}\frac{\partial}{i\partial x_l}\right)^2$$

and let P_x^q be its qth successive iterate, where q is a non-negative integer. We shall now apply the operator P_x^q to the Fourier series in (3.8) and then use Lemma 5 on the summability of the derived series. Before doing so, we observe that our function $f(Tx_1,\dots,Tx_\kappa)$

which is evaluated in (3.11) is analytic in $(x_1,\dots,x_\kappa)$ for every ξ which is such that

$$\xi^2 \neq Q(h+l,h+l) \tag{3.13}$$

for all lattice-points l. We thus obtain the following

THEOREM II. *If ξ satisfies the conditions* (3.13), *and $\beta > -1$, then the series*

$$\sum_{m=1}^{\infty} \frac{r_\kappa^P(\lambda_m,h)}{\lambda_m^{\frac{1}{2}\beta+\frac{1}{4}\kappa}} J_{\beta+\frac{1}{2}\kappa}(2\pi\xi\sqrt{\lambda_m})\lambda_m^q \tag{3.14}$$

is summable $(R;\lambda,\gamma)$ for $\gamma > \frac{1}{2}(\kappa-1)+2q$, where q is a non-negative integer.

From (3.14) it is clear that

$$\sum_{m=1}^{\infty} r_\kappa^P(\lambda_m,h) J_{\beta+\frac{1}{2}\kappa}(2\pi\xi\sqrt{\lambda_m})\lambda_m^s \tag{3.15}$$

is summable $(R;\lambda,\gamma)$ for $\gamma > \frac{1}{2}(\kappa-1)+2q$, where $s = q-\frac{1}{2}\beta-\frac{1}{4}\kappa$ and $q = 0, 1, 2,\dots$.

However, by Lemma 4, the abscissa of summability-(λ,γ) is a convex function of γ and hence (3.15) holds for *every* $q \geqslant 0$ integral or not. Thus we have

THEOREM III. *If ξ satisfies* (3.13), *then the series*

$$\sum_{m=1}^{\infty} r_\kappa^P(\lambda_m,h) J_{\beta+\frac{1}{2}\kappa}(2\pi\xi\sqrt{\lambda_m})\lambda_m^s \tag{3.16}$$

is summable $(R;\lambda,\gamma)$ for $\gamma > \frac{1}{2}(\kappa-1)$ and $s < \frac{1}{4}-\frac{1}{2}\beta-\frac{1}{2}\kappa+\frac{1}{2}\gamma$ for every $\beta > -1$.

If we now write $\rho = \beta+\frac{1}{2}\kappa$ and choose ϵ such that $0 < \epsilon < 1$, then from (3.16) we can easily conclude that

$$\sum_{m=1}^{\infty} r_\kappa^P(\lambda_m,h) J_\rho(2\pi\xi\sqrt{\lambda_m})\lambda_m^s \tag{3.17}$$

is summable $(R;\lambda,\gamma)$ for $\gamma > \frac{1}{2}(\kappa-1)$, $s < \frac{1}{4}-\frac{1}{2}\rho-\frac{1}{4}\kappa+\frac{1}{2}\gamma$ and also

$$\sum_{m=1}^{\infty} r_\kappa^P(\lambda_m,h) J_{\rho+\epsilon}(2\pi\xi\sqrt{\lambda_m})\lambda_m^s \tag{3.18}$$

is summable $(R;\lambda,\gamma)$ for the same $\gamma > \frac{1}{2}(\kappa-1)$ and

$$s < \tfrac{1}{4}-\tfrac{1}{2}\rho-\tfrac{1}{4}\kappa+\tfrac{1}{2}\gamma.$$

Now by Lemma 2 we can combine the Bessel functions $J_\rho(x)$ and $J_{\rho+\epsilon}(x)$ to give $x^{-\frac{1}{2}}e^{ix}$ for $x = 2\pi\xi\sqrt{\lambda_n}$. Further, it is possible to choose m so large that

$$\sum r_\kappa^P(\lambda_n,h)\lambda_n^s O(\sqrt{\lambda_n}^{-m-\frac{1}{2}}), \tag{3.19}$$

which is $$O\left(\sum_{n=0}^{\infty} \frac{1}{\{P(n,n)\}^{-s+\frac{1}{4}m+\frac{1}{4}}}\right),$$

is absolutely convergent. Having chosen such an m, by a standard property of Riesz summability, we observe that

$$\sum r_\kappa^P(\lambda_n, h)\lambda_n^s \frac{J_\eta(2\pi\xi\sqrt{\lambda_n})}{(\sqrt{\lambda_n})^\delta} \tag{3.20}$$

is also summable $(R;\lambda,\gamma)$ for $\gamma > \frac{1}{2}(\kappa-1)$, $s < \frac{1}{4}-\frac{1}{2}\rho-\frac{1}{4}\kappa+\frac{1}{2}\gamma$, and for $\delta = 0, 1, \ldots, m-1$ and $\eta = \rho$ or $\rho+\epsilon$.

Combining (3.17), (3.18), (3.19), and (3.20), and using Lemma 2, we obtain the following

THEOREM IV. *If ξ satisfies* (3.13), *then the series*

$$\sum_{m=1}^{\infty} r_\kappa^P(\lambda_m, h)e^{2\pi i\xi\sqrt{\lambda_m}}\lambda_m^s \tag{3.21}$$

is summable $(R;\lambda,\gamma)$ for $\gamma > \frac{1}{2}(\kappa-1)$ and $s < \frac{1}{2}-\frac{1}{2}\kappa+\frac{1}{2}\gamma$.

4. Bessel series with sums zero

We shall draw some consequences of special interest from Theorem II. The sum of the series in (3.14) can, in general, be computed by differentiation of the function $f(Tx_1, \ldots, Tx_\kappa)$; there are special cases in which the sum is zero, for example, if $\beta = 0$ and $q = 1, 2, 3, \ldots$; or, if $\beta = 1$ and $q = 2, 3, 4, \ldots$; or, if $\beta = 2$ and $q = 3, 4, \ldots$. Thus we have the following

THEOREM V. *The series*

$$\sum \frac{r_\kappa^P(\lambda_m, h)}{\lambda_m^{\frac{1}{4}\beta+\frac{1}{4}\kappa-q}} J_{\beta+\frac{1}{2}\kappa}(2\pi\xi\sqrt{\lambda_m}) \tag{4.1}$$

is summable $(R;\lambda,\gamma)$ for $\gamma > \frac{1}{2}(\kappa-1)+2q$ to the sum zero for every pair of positive integral values of the type $\beta = n$ and $q = n+1, n+2, \ldots$, where $n = 0, 1, \ldots$.

Choosing $\beta = 0$ in (4.1) we observe that

$$\sum r_\kappa^P(\lambda_m, h)\frac{J_{\frac{1}{2}\kappa}(2\pi\xi\sqrt{\lambda_m})}{\sqrt{\lambda_m}}\lambda_m^{q-\frac{1}{4}\kappa+\frac{1}{4}} \tag{4.2}$$

is summable to zero, for $q = 1, 2, 3, \ldots$. Again choosing $\beta = 1$ (and writing $q+1$ for q) we see that

$$\sum r_\kappa^P(\lambda_m, h)J_{\frac{1}{2}\kappa+1}(2\pi\xi\sqrt{\lambda_m})\lambda_m^{q-\frac{1}{4}\kappa+\frac{1}{4}} \tag{4.3}$$

is also summable to zero, for $q = 1, 2, 3, \ldots$.

Utilizing the relation

$$2n\frac{J_n(x)}{x} = J_{n-1}(x)+J_{n+1}(x),$$

and combining (4.2) and (4.3), we obtain that

$$\sum r_\kappa^P(\lambda_m, h)J_{\frac{1}{2}\kappa-1}(2\pi\xi\sqrt{\lambda_m})\lambda_m^{q-\frac{1}{4}\kappa+\frac{1}{4}} \tag{4.4}$$

is summable to zero, for $q = 1, 2, 3, \ldots$.

To determine the order of summability of (4.2) and (4.4) we have only to apply Theorem III, with $s = q-\frac{1}{4}\kappa$ and $s = q-\frac{1}{4}\kappa+\frac{1}{2}$ respectively. We then conclude that (4.2) is summable $(R;\lambda,\gamma)$ for

$$\gamma > 2q+\tfrac{1}{2}(\kappa-3), \qquad q = 1, 2, \ldots$$

and (4.4) is summable $(R;\lambda,\gamma)$ for $\gamma > 2q+\frac{1}{2}(\kappa-1)$. Setting $\kappa = 2$ we obtain the following

THEOREM VI. (A) *If ξ satisfies* (3.13), *then the series*

$$\sum_{m=1}^{\infty} r_2^P(\lambda_m, h)J_0(2\pi\xi\sqrt{\lambda_m})\lambda_m^q$$

is summable $(R;\lambda,\gamma)$ for $\gamma > 2q+\frac{1}{2}$ to the sum zero, for every positive integer q.

(B) *If ξ satisfies* (3.13), *then the series*

$$\sum_{m=1}^{\infty} r_2^P(\lambda_m, h)J_1(2\pi\xi\sqrt{\lambda_m})\lambda_m^q$$

is summable $(R;\lambda,\gamma)$ for $\gamma > 2q+\frac{1}{2}$ to the sum zero, for every q which is half of an odd positive integer.

This includes a theorem of Dixon and Ferrar [4, II, Th. III (A)].

5. Unimodular forms

Hereafter we shall assume the forms $P(n,n)$ and $Q(n,n)$ to be integer-valued so that $\lambda_n = n$, say. In this case, we shall see that Theorem IV can be strengthened by relaxation of the restriction on γ. We shall be able, in some cases, to deduce actual convergence of the series (3.21) with $\lambda_n = n$, by the application of Lemma 3. Suppose that

$$\sum r_\kappa(n,h)e^{2\pi i\xi\sqrt{n}}n^p \equiv \sum A_n.$$

Then, $\sum A_n$ is summable $(R;n,\gamma)$ for $\gamma > \frac{1}{2}(\kappa-1)$ and $p < \frac{1}{2}(1-\kappa+\gamma)$. Also

$$A_n = O(n^{\frac{1}{4}(\kappa-2)+p+\frac{1}{4}\epsilon}) \quad (\epsilon > 0).$$

Hence, by Lemma 3, $\sum A_n/n^\sigma$ is summable $(R;n,\eta)$ for $0 \leqslant \eta < \gamma$ and

$$\sigma > \tfrac{1}{2}(\kappa+2p+\epsilon)\left(\frac{\gamma-\eta}{\gamma+1}\right).$$

Thus

$$-\sigma < \left(\frac{\eta-\gamma}{\gamma+1}\right)\left(\frac{\kappa+\epsilon}{2}+p\right) < \left(\frac{\eta-\gamma}{\gamma+1}\right)\left(\frac{\gamma+1+\epsilon}{2}\right) = \frac{\eta-\gamma}{2}+\frac{\epsilon}{2(\gamma+1)}$$

and

$$p-\sigma < \tfrac{1}{2}(1-\kappa+\eta).$$

Hence we have the following

THEOREM VII. *If ξ^2 is non-integral,*

$$\sum_{n=1}^{\infty} r_\kappa(n,h)e^{2\pi i\xi\sqrt{n}}n^l$$

is summable $(R;n,\eta)$ for $\eta \geqslant 0$ and $l < \frac{1}{2}(1-\kappa+\eta)$.

This includes a theorem of Dixon and Ferrar [4, II, Th. III (B)]. We may remark that from Theorem VII it follows that

$$\sum r_\kappa(n,h)J_\mu(2\pi\xi\sqrt{n})n^l$$

is summable $(R;n,\eta)$ for $\eta \geqslant 0$ and $l < \frac{3}{4}-\frac{1}{2}\kappa+\frac{1}{2}\eta$, $\mu > -1$.

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ON THE DISTRIBUTION OF THE SEQUENCE $n^2\theta \pmod 1$

By H. HEILBRONN (*Bristol*)

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1. THE object of this paper is to prove the

THEOREM. *For every integer $N \geqslant 1$ and every real θ, integers n and g can be found such that*

$$1 \leqslant n \leqslant N, \quad |n^2\theta - g| \leqslant c(\eta)N^{-\frac{1}{2}+\eta}, \tag{1.1}$$

where η is an arbitrarily small positive number and where $c(\eta)$ depends on η only.

The essential fact is that the inequality (1.1) holds uniformly in θ.

A corresponding result was proved by Vinogradov† with the upper bound $N^{-\frac{3}{5}+\eta}$ instead of $N^{-\frac{1}{2}+\eta}$. It is not impossible that the theorem remains valid if the upper bound is replaced even by $N^{-1+\eta}$, but a considerable improvement of (1.1) appears to be unattainable at present.

My proof of the theorem is not very different from Vinogradov's proof, but a slight change in the fundamental technique enables me to make a more delicate analysis of the error terms which arise.

I wish to thank Mr. H. Todd who pointed out to me the earlier history‡ of this problem and thereby started me on this investigation.

2. N, θ, and η have always the meanings given above. Without loss of generality we assume that θ is irrational. The symbol O implies a constant which depends on η only. We write O_η as an abbreviation for $O(N^\eta)$. As η is arbitrary, the relation

$$O_\eta O_\eta = O_\eta$$

may be applied in a bounded number of places.

Throughout the paper small and capital italics other than e, i, and O denote integers. With the further exception of g, b, a, and A all these integers will be positive.

† *Bull. Acad. Sci. U.R.S.S.* (6) 27 (1927), 567–78.

‡ For further references see J. F. Koksma, *Diophantische Approximationen*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 4 (1936), Kapitel X, § 1, 2.

We define $\epsilon = \epsilon(N, \theta)$ by the relation

$$\epsilon = \min_{\substack{1 \leqslant n \leqslant N \\ g}} |n^2\theta - g|,$$

so that $0 < \epsilon < \frac{1}{2}$. We assume at once that $\epsilon > N^{-\frac{1}{2}}$.

The function $\phi_\epsilon(\xi)$ has period one in ξ and is defined for $-\frac{1}{2} < \xi \leqslant \frac{1}{2}$ by

$$\phi_\epsilon(\xi) = \max(0, 1 - \epsilon^{-1}|\xi|).$$

It is easily seen that $\phi_\epsilon(\xi)$ has the Fourier expansion

$$\phi_\epsilon(\xi) = \epsilon + 2\pi^{-2}\epsilon^{-1} \sum_{m=1}^{\infty} m^{-2} \sin^2(\pi m\epsilon)\cos(2\pi m\xi). \tag{2.1}$$

LEMMA 1. *For every m we can find four integers a_m, q_m, A_m, Q_m such that*

$$q_m \leqslant N < Q_m, \qquad (a_m, q_m) = 1, \qquad (A_m, Q_m) = 1, \tag{2.2}$$

$$|m\theta - a_m/q_m| < q_m^{-1}Q_m^{-1}, \tag{2.3}$$

$$|m\theta - A_m/Q_m| < Q_m^{-2}. \tag{2.4}$$

Proof. Let a_m/q_m be the last convergent of the continued fraction for $m\theta$ whose denominator does not exceed N, and let A_m/Q_m be the first convergent whose denominator does exceed N. The lemma then follows at once from the theory of continued fractions.

LEMMA 2.

$$\sum_{n=1}^{N} e^{2\pi i n^2 m\theta} = O_\eta(Nq_m^{-\frac{1}{2}}),$$

$$\sum_{n=1}^{N} e^{2\pi i n^2 m\theta} = O_\eta(Q_m^{\frac{1}{2}}).$$

Proof. It has been well known for a long time† that, if

$$(a, q) = 1, \qquad \alpha = a/q + \beta, \qquad |\beta| \leqslant N^{-1}q^{-1}, \tag{2.5}$$

then

$$\sum_{n=1}^{N} e^{2\pi i n^2 \alpha} = O(Nq^{-\frac{1}{2}+\eta} + q^{\frac{1}{2}+\eta}). \tag{2.6}$$

If we put $\alpha = m\theta$, $a = a_m$, $q = q_m$, then, by (2.3) and (2.2),

$$|\beta| < q_m^{-1}Q_m^{-1} < q_m^{-1}N^{-1}.$$

Thus (2.5) holds and (2.6) gives the first part of the lemma since, by (2.2),

$$Nq_m^{-\frac{1}{2}+\eta} + q_m^{\frac{1}{2}+\eta} \leqslant 2N^{1+\eta}q_m^{-\frac{1}{2}}.$$

† See Davenport and Heilbronn, *J. of London Math. Soc.* 21 (1946), 185–93, Lemma 6.

If we put $\alpha = m\theta$, $a = A_m$, $q = Q_m$, then, by (2.4) and (2.2),

$$|\beta| < Q_m^{-2} < Q_m^{-1}N^{-1}.$$

Again, (2.5) holds and (2.6) gives the second part of the lemma since, for $Q_m \leqslant N^2$,

$$NQ_m^{-\frac{1}{2}+\eta} + Q_m^{\frac{1}{2}+\eta} \leqslant 2N^{2\eta}Q_m^{\frac{1}{2}},$$

whereas for $Q_m > N^2$ the second part of the lemma is trivial.

LEMMA 3.

$$1 = O_\eta\Big\{\sum_{m\leqslant\epsilon^{-1}} \min(q_m^{-\frac{1}{2}}, N^{-1}Q_m^{\frac{1}{2}}) + \epsilon^{-2}\sum_{m>\epsilon^{-1}} m^{-2}\min(q_m^{-\frac{1}{2}}, N^{-1}Q_m^{\frac{1}{2}})\Big\}.$$

Proof. It follows from the definition of ϵ and from (2.1) that

$$0 = \sum_{n=1}^{N}\phi_\epsilon(n^2\theta) = \sum_{n=1}^{N}\Big\{\epsilon + 2\pi^{-2}\epsilon^{-1}\sum_{m=1}^{\infty} m^{-2}\sin^2(\pi m\epsilon)\cos(2\pi mn^2\theta)\Big\}$$

$$= N\epsilon + 2\pi^{-2}\epsilon^{-1}\sum_{m=1}^{\infty} m^{-2}\sin^2(\pi m\epsilon)\sum_{n=1}^{N}\cos(2\pi n^2 m\theta).$$

Hence, dividing by $N\epsilon$,

$$1 \leqslant N^{-1}\epsilon^{-2}\sum_{m=1}^{\infty} m^{-2}\sin^2(\pi m\epsilon)\Big|\sum_{n=1}^{N} e^{2\pi i n^2 m\theta}\Big|,$$

and the result follows from Lemma 2 if we use the inequalities

$$\sin^2(\pi m\epsilon) \leqslant \pi^2 m^2\epsilon^2 \quad \text{for} \quad m \leqslant \epsilon^{-1},$$

$$\sin^2(\pi m\epsilon) \leqslant 1 \quad \text{for} \quad m > \epsilon^{-1}.$$

LEMMA 4. If $|\theta - b/r| = \delta$, $1 \leqslant r \leqslant N$, then

$$\epsilon \leqslant r^2\delta.$$

Proof. $$|r^2\theta - rb| = r^2\delta.$$

LEMMA 5. *For $\omega \geqslant 1$, $\tau > 0$ let $\Lambda(\omega, \tau)$ denote the number of solutions of the inequality*

$$|\theta - b/r| \leqslant \tau$$

subject to $$r \leqslant \omega, \qquad (b, r) = 1.$$

Then $$\Lambda(\omega, \tau) \leqslant 1 + 2\omega^2\tau.$$

Proof. Any two different pairs b/r, b'/r' have at least the distance

$$1/(rr') \geqslant \omega^{-2}$$

from each other. So there can be at most $1 + 2\omega^2\tau$ such fractions in an interval of length 2τ.

So far the work has been routine, and has differed from Vinogradov's proof only by the introduction of the function $\phi_\epsilon(\xi)$ whereas Vinogradov employs the function

$$\psi_\epsilon(\xi) = \begin{cases} 1 & (|\xi| < \epsilon), \\ 0 & (\epsilon < |\xi| \leqslant \frac{1}{2}), \end{cases}$$

which has a more slowly convergent Fourier series.

3. I now introduce four more integers l, b, r, d which will depend on m (and N and θ as well). I put

$$l = (a_m, mq_m), \tag{3.1}$$

$$a_m = lb, \tag{3.2}$$

$$mq_m = lr, \tag{3.3}$$

so that $$(b, r) = 1.$$

Since $l \mid a_m$ and $(a_m, q_m) = 1$, we have $(l, q_m) = 1$; this and (3.1) give $l \mid m$ and we can put

$$m = ld, \tag{3.4}$$

so that by (3.3)

$$r = q_m d. \tag{3.5}$$

Finally we put

$$|\theta - b/r| = \delta. \tag{3.6}$$

I show first that b is uniquely determined by r. Translated into the new notation (2.3) gives

$$|ld\theta - lbdr^{-1}| < d^2 r^{-2} q_m Q_m^{-1},$$

$$|r\theta - b| < dr^{-1} l^{-1} q_m Q_m^{-1} \leqslant (d/r)(q_m/Q_m). \tag{3.7}$$

Since $d \mid r$, we have either $d \leqslant \frac{1}{2}r$,

$$|r\theta - b| < \tfrac{1}{2}, \tag{3.8}$$

or $d = r$, $q_m = 1$, $Q_m \geqslant 2$, and (3.7) again implies (3.8). In either case (3.8) is true, and b depends on r (and θ) only.

The object is to transform Lemma 3 so that instead of summing over m we sum over the three variables r, d, l. In this summation we shall treat r, d, l as independent variables subject to certain restrictions which I am now going to develop.

We note first that by (3.4), (3.2), and (3.5)

$$|m\theta - a_m/q_m| = ld|\theta - b/r| = dl\delta. \tag{3.9}$$

The condition $q_m \leqslant N$ gives

$$rd^{-1} \leqslant N. \tag{3.10}$$

The condition $Q_m > N$ and (2.3) together with (3.5) and (3.9) give

$$N < Q_m < q_m^{-1}|m\theta - a_m/q_m|^{-1} = r^{-1}l^{-1}\delta^{-1} \tag{3.11}$$

and

$$l < N^{-1}r^{-1}\delta^{-1}. \tag{3.12}$$

We restrict m further by the additional condition

$$mq_m < N\epsilon^{-1}. \tag{3.13}$$

The introduction of (3.13) will be justified later. From (3.13) and (3.3) we have

$$l < Nr^{-1}\epsilon^{-1}. \tag{3.14}$$

Since $l \geqslant 1$, we deduce from (3.12) and (3.14)

$$\delta < N^{-1}r^{-1}, \tag{3.15}$$

$$r < N\epsilon^{-1}. \tag{3.16}$$

To these conditions we add by virtue of Lemma 4

$$\delta \geqslant r^{-2}\epsilon \quad \text{for} \quad r \leqslant N. \tag{3.17}$$

4. The first step is the transformation of Lemma 3.

LEMMA 6.

$$1 = O_\eta(N^{-\frac{1}{2}}\epsilon^{-1}) + O_\eta\Big[\sum_{r<N\epsilon^{-1}} \sum_{\substack{d|r \\ d\geqslant N^{-1}r}} \sum_{l} \min\{1, (dl\epsilon)^{-2}\}\min(r^{-\frac{1}{2}}d^{\frac{1}{2}}, N^{-1}r^{-\frac{1}{2}}l^{-\frac{1}{2}}\delta^{-\frac{1}{2}})\Big],$$

where the summation is restricted by the inequalities (3.12), (3.14).

Proof. Our first step is the justification of the conditions (3.13). The sum in Lemma 3 extended over those m for which (3.13) is untrue does not exceed

$$O_\eta\Big(\sum_{m\leqslant\epsilon^{-1}} N^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}m^{\frac{1}{2}} + \epsilon^{-2}\sum_{m>\epsilon^{-1}} m^{-2}N^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}m^{\frac{1}{2}}\Big)$$
$$= O_\eta\{N^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}(\epsilon^{-1})^{\frac{3}{2}} + N^{-\frac{1}{2}}\epsilon^{-\frac{3}{2}}(\epsilon^{-1})^{-\frac{1}{2}}\}$$
$$= O_\eta(N^{-\frac{1}{2}}\epsilon^{-1}).$$

This calculation justifies the restriction of the summation by (3.13).

The restrictions in the summation explicitly mentioned in the formula of Lemma 6 are satisfied by virtue of (3.16), (3.5), and (3.10). So all we have to do is to show that each term in the sum of Lemma 3 is majorized by the corresponding term in the sum of Lemma 6.

By (3.4)

$$1 \leqslant \min\{1, (\epsilon m)^{-2}\} = \min\{1, (dl\epsilon)^{-2}\} \quad \text{for} \quad m \leqslant \epsilon^{-1},$$
$$\epsilon^{-2}m^{-2} \leqslant \min\{1, (\epsilon m)^{-2}\} = \min\{1, (dl\epsilon)^{-2}\} \quad \text{for} \quad m > \epsilon^{-1}.$$

By (3.5) $$q_m^{-\frac{1}{2}} = r^{-\frac{1}{2}} d^{\frac{1}{2}}.$$

By (3.11) $$N^{-1} Q_m^{\frac{1}{2}} < N^{-1} r^{-\frac{1}{2}} l^{-\frac{1}{2}} \delta^{-\frac{1}{2}}.$$

This concludes the proof of Lemma 6.

We now split the sum over r in Lemma 6 into three sums

$$\sum_r = \sum_\alpha + \sum_\beta + \sum_\gamma$$

according to the conditions

(α) $$r \leqslant N;$$

(β) $$r > N, \qquad \delta \geqslant N^{-2}\epsilon;$$

(γ) $$r > N, \qquad \delta < N^{-2}\epsilon.$$

LEMMA 7. $$\sum_\alpha = O_\eta(N^{-\frac{1}{2}}\epsilon^{-1}).$$

Proof. By (3.12),

$$\begin{aligned} \sum_\alpha &\leqslant \sum_{r \leqslant N} \sum_{d \mid r} \sum_{l < N^{-1} r^{-1} \delta^{-1}} N^{-1} r^{-\frac{1}{2}} l^{-\frac{1}{2}} \delta^{-\frac{1}{2}} \\ &= N^{-1} \sum_{r \leqslant N} r^{-\frac{1}{2}} \delta^{-\frac{1}{2}} \sum_{d \mid r} \sum_{l < N^{-1} r^{-1} \delta^{-1}} l^{-\frac{1}{2}} \\ &= O_\eta\Big(N^{-\frac{3}{2}} \sum_{r \leqslant N} r^{-1} \delta^{-1}\Big). \end{aligned} \tag{4.1}$$

Then (4.1) and (3.17) give

$$\begin{aligned} \sum_\alpha &= O_\eta\Big(N^{-\frac{3}{2}} \sum_{r \leqslant N} r\epsilon^{-1}\Big) \\ &= O_\eta\Big(N^{-\frac{1}{2}}\epsilon^{-1} \sum_{r \leqslant N} 1\Big). \end{aligned}$$

But this sum contains at most two terms owing to the condition (3.15). For otherwise we should have three fractions

$$b_j/r_j \quad (j = 1, 2, 3)$$

such that

$$b_1/r_1 < b_2/r_2 < b_3/r_3,$$

$$|\theta - b_j/r_j| < N^{-1} r_j^{-1} \quad \text{for} \quad j = 1, 2, 3.$$

Hence either

$$\theta - b_2/r_2 > 0 \quad \text{or} \quad \theta - b_2/r_2 < 0,$$

$$0 < b_2/r_2 - b_1/r_1 < \theta - b_1/r_1 \quad \text{or} \quad 0 < b_3/r_3 - b_2/r_2 < b_3/r_3 - \theta,$$

$$0 < b_2/r_2 - b_1/r_1 < N^{-1} r_1^{-1} \quad \text{or} \quad 0 < b_3/r_3 - b_2/r_2 < N^{-1} r_3^{-1},$$

$$1 \leqslant b_2 r_1 - b_1 r_2 < N^{-1} r_2 \quad \text{or} \quad 1 \leqslant b_3 r_2 - b_2 r_3 < N^{-1} r_2,$$

$$r_2 > N$$

which is a contradiction.

LEMMA 8. $$\sum_{\beta} = O_\eta(N^{-\frac{1}{2}}\epsilon^{-1}).$$

Proof. Repeating *mutatis mutandis* the calculation in the proof of Lemma 7 leading to (4.1) we obtain

$$\sum_{\beta} = O_\eta\Big(N^{-\frac{3}{2}} \sum_{N<r<N\epsilon^{-1}} r^{-1}\delta^{-1}\Big) \tag{4.2}$$

where, by (3.15) and (β),

$$N^{-2}\epsilon \leqslant \delta < N^{-2}.$$

We can cover the rectangle

$$N \leqslant r \leqslant N\epsilon^{-1}, \qquad N^{-2}\epsilon \leqslant \delta \leqslant N^{-2}$$

by $O(\log^2\epsilon^{-1})$ rectangles

$$\Omega \leqslant r \leqslant 2\Omega, \qquad \Delta \leqslant \delta \leqslant 2\Delta$$

allowing for overlapping such that

$$N \leqslant \Omega < N\epsilon^{-1}, \qquad N^{-2}\epsilon \leqslant \Delta < N^{-2}.$$

Hence, by (4.2) and Lemma 5,

$$\begin{aligned}
\sum_{\beta} &= O_\eta\Big[N^{-\frac{3}{2}} \log^2\epsilon^{-1} \max_{\Omega,\Delta}\{\Omega^{-1}\Delta^{-1}\Lambda(2\Omega, 2\Delta)\}\Big] \\
&= O_\eta\Big[N^{-\frac{3}{2}} \max_{\Omega,\Delta}\{\Omega^{-1}\Delta^{-1}(1+16\Omega^2\Delta)\}\Big] \\
&= O_\eta\Big\{N^{-\frac{3}{2}}(\max_{\Omega,\Delta} \Omega^{-1}\Delta^{-1} + 16 \max_{\Omega} \Omega)\Big\} \\
&= O_\eta\{N^{-\frac{3}{2}}(N\epsilon^{-1} + 16N\epsilon^{-1})\} \\
&= O_\eta(N^{-\frac{1}{2}}\epsilon^{-1}).
\end{aligned}$$

LEMMA 9. $$\sum_{\gamma} = O_\eta(N^{-\frac{1}{2}}\epsilon^{-1}).$$

Proof.

$$\begin{aligned}
\sum_{\gamma} &\leqslant \sum_{\substack{N\leqslant r<N\epsilon^{-1} \\ \delta<N^{-2}\epsilon}} \sum_{\substack{d\mid r \\ d>N^{-1}r}} \sum_{l=1}^{\infty} \min\{1, (dl\epsilon)^{-2}\} r^{-\frac{1}{2}} d^{\frac{1}{2}} \\
&= O\Big\{\sum_{\substack{N\leqslant r<N\epsilon^{-1} \\ \delta<N^{-2}\epsilon}} r^{-\frac{1}{2}} \sum_{\substack{d\mid r \\ d>N^{-1}r}} d^{\frac{1}{2}}(d\epsilon)^{-1}\Big\} \\
&= O\Big(\epsilon^{-1} \sum_{\substack{N\leqslant r<N\epsilon^{-1} \\ \delta<N^{-2}\epsilon}} r^{-\frac{1}{2}} \sum_{\substack{d\mid r \\ d>N^{-1}r}} d^{-\frac{1}{2}}\Big) \\
&= O_\eta\Big(N^{\frac{1}{2}}\epsilon^{-1} \sum_{\substack{N\leqslant r<N\epsilon^{-1} \\ \delta<N^{-2}\epsilon}} r^{-1}\Big) \\
&= O_\eta\Big\{N^{\frac{1}{2}}\epsilon^{-1} \int_{N}^{N\epsilon^{-1}} \omega^{-1}\, d\Lambda(\omega, N^{-2}\epsilon)\Big\}.
\end{aligned} \tag{4.3}$$

By Lemma 5

$$\int_N^{N\epsilon^{-1}} \omega^{-1}\,d\Lambda(\omega, N^{-2}\epsilon) \leqslant (N\epsilon^{-1})^{-1}\Lambda(N\epsilon^{-1}, N^{-2}\epsilon) + \int_N^{N\epsilon^{-1}} \omega^{-2}\Lambda(\omega, N^{-2}\epsilon)\,d\omega$$

$$\leqslant N^{-1}\epsilon(1+2\epsilon^{-1}) + \int_N^{N\epsilon^{-1}} \omega^{-2}(1+2\omega^2 N^{-2}\epsilon)\,d\omega$$

$$\leqslant N^{-1}\epsilon + 2N^{-1} + N^{-1} + 2(N\epsilon^{-1})(N^{-2}\epsilon) < 6N^{-1}.$$

This inequality and (4.3) prove Lemma 9.

Proof of the theorem. Lemmas 6, 7, 8, 9 give

$$1 = O_\eta(N^{-\frac{1}{2}}\epsilon^{-1}),$$

$$\epsilon = O_\eta(N^{-\frac{1}{2}}),$$

which is the theorem.

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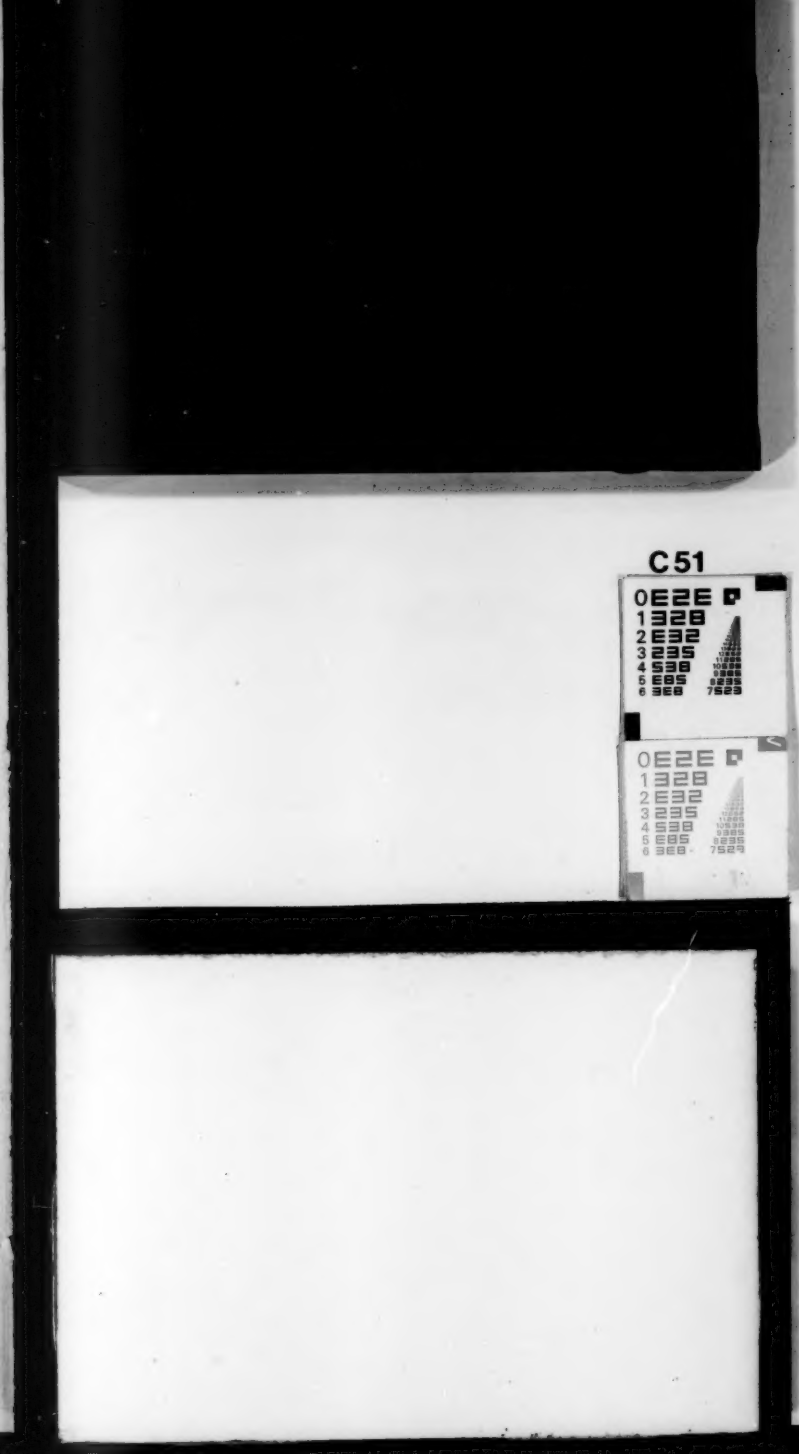
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